



EQUILIBRIUM MARKET PRICES OF RISKS AND RISK AVERSION IN A COMPLETE STOCHASTIC VOLATILITY MODEL WITH HABIT FORMATION: EMPIRICAL RISK AVERSION FROM S&P 500 INDEX OPTIONS

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EMPIRICAL RISK AVERSION FROM S&P 500 INDEX OPTIONS

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Considering a pure exchange economy with habit formation utility, the theoretical part of this dissertation explores the equilibrium relationships between the market pricing kernel, the market prices of risks and the market risk aversion under a continuous time stochastic volatility model completed by liquidly traded put options. We demonstrate with these equilibrium relations that the risk neutral pricing partial differential equation is a restricted version of the fundamental pricing equation provided in Garman (1976). We also show that in this completed market stochastic volatility cannot explain the documented empirical pricing kernel puzzle (Jackwerth (2000)). Instead, a habit formation utility offers a possible explanation of the puzzle. The derived quantitative relation between the market prices of risks and the market risk aversion also provides a new way to extract empirical market risk aversion. Based upon this theoretical relation between market prices of risks and the market risk aversion in a Heston model, we empirically extract the market prices of risks and risk aversion from the options market using cross-sectional fitting. Specifically we consider a restricted model where only the volatility risk is allowed to freely change and an unrestricted model where all model parameters are allowed to freely change. For the restricted model, we determine other parameters by Efficient Method of Moments (EMM). Using European call options data, we find an implied risk aversion

smile, indicating that individual groups of investors trading options with different strike prices have different risk aversions. We also extracted an average or aggregated market risk aversion by minimizing the mean squared pricing error across all strikes. This represents the risk aversion level for the whole market in the sense of “averaging”. None of these risk aversions are negative across moneyness, hence indicating that adding stochastic volatility to the model will not reproduce the documented pricing kernel puzzle. In addition, the market price of volatility risk is small in values compared with the market price of asset risk, implying that the major driving factor of market risk aversion and pricing kernel is the asset risk. This is consistent with the sensitivity analysis conducted on the option prices with respect to the market prices of risks. For the unrestricted model, we observe similar behavior for the two market prices of risks using a different data set, S&P500 index futures options. We find that the asset risk and volatility risk premium generally move opposite across the strikes. The variation of volatility risk decreases and the absolute values converge to zero with longer time to maturity. So the asset risk dominates the pricing more for options with longer maturities.

BIOGRAPHICAL SKETCH

Qian Han got his Bachelor of International Economics in July 1998 from Harbin Engineering University, China P.R. In May 2003 he got Master of Public Affairs from Cornell Institute of Public Affairs at Cornell University. He is now a Ph.D. candidate at the department of Applied Economics and Management (AEM) at Cornell University. His doctoral major is applied finance and economics.

Dedicated to my parents, my wife and my son

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I also thank my advisor, Professor Turvey, for his suggestion of the dissertation topic to me two years ago and his very insightful advices on how to approach the problem. Especially, when I got lost in the computer programmings and models, he timely brought me back to track and kept me stay focused on the real issue this dissertation should consider.

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LIST OF ABBREVIATIONS

ABm: Arithmetic Brownian motion
AMRA: average market risk aversion
ATM: at the money
CEV: Constant Elasticity Volatility
EMM: efficient method of moments
EPK: empirical pricing kernel
FDM: finite difference method
FEM: finite element method
GBm: Geometric Brownian motion
IMRA: individual market risk aversion
ITM: in the money
MPR: market price of risk
MRA: market risk aversion
NM: Nelder-Mead
OTM: out of the money
PDE: partial differential equation
QEM: quadratic exponential with martingale correction
SVM: stochastic volatility model
TMRA: theoretical market risk aversion

LIST OF SYMBOLS

Ω : an outcome space

\mathcal{F} : the sigma algebra corresponding to the outcome space Ω

P : a probability measure assigned on (Ω, \mathcal{F})

μ : drift rate in a stochastic volatility model

σ : volatility of the underlying asset

ξ : volatility of volatility in a stochastic volatility model

v : initial volatility of a stochastic volatility model

ρ : correlation coefficient in a stochastic volatility model

ς : the stochastic volatility factor in a stochastic volatility model

τ : the elasticity of option price with respect to the underlying asset

ψ : the parameter set for optimization problem

ω : a typical element in the outcome space Ω

κ : mean reversion rate in a stochastic volatility model

λ : market price of risk

γ : market risk aversion

δ : the dividend rate

ζ : the Radon-Nikodym process

Γ : the payoff space

Q : the Q -measure (risk neutral measure)

G : the market pricing kernel

PREFACE

When I came to Professor Turvey's office for a possible Ph.D dissertation topic, he suggested that I look at the real economic meaning of the market price of risk. At that time, I happened to have just finished a financial engineering course where the concept was introduced. I was surprised to find that this concept appeared in both traditional asset pricing theory and modern financial engineering literature. There must be some inner connections that unify this concept of market price of risk. There Professor Turvey's suggestion immediately interested me and he identified several references including his 2006 paper on empirical market price of risk for the live cattle futures options market.

Under the guidance of Professor Turvey and many helps from others in my committee members, I developed a theoretical model of pure exchange economy combined with a stochastic volatility model with habit formation utility to examine the relationship between the market prices of risks and the market risk aversion in equilibrium. Built upon this theoretical result, I look at the empirical risk premium and market risk aversion in the S&P 500 index options market. Interesting results are found about the behavior of implied risk aversion across the strikes. Professor Turvey and I believe that these results, both theoretical and empirical, shed some lights on what the concept of market price of risk really means and how the market investors, both individually and as a whole, perceive about the risks embodied in the index options market.

In chapter 1, I introduce the central topic of the dissertation and do a detailed literature review on risk aversion, prices of risks and the pricing kernel. Several key issues that are addressed in the dissertation are identified. In chapter 2, I present the theoretical model of equilibrium relationships between the market prices of risks, the

risk aversion and the pricing kernel. This serves as the basis for the empirical studies in later chapters. The idea of finding the relation between the market pricing kernel and the market price of risk comes from Garman's paper in 1976. To my surprise, I found that if one substitutes the pricing kernel term in his fundamental equation with the price of risk term, then one ends up with the pricing partial differential equation now popular in many literatures. With this observation, I started considering when this relation holds. One of the possible answers is the model in this dissertation. On suggestion of one of my friends, Zhaogang Song, I later extend the analysis to include a habit utility for the representative agent, which offers a possible explanation to the documented pricing kernel puzzle by Jackwerth (2000).

The theoretical results presented in chapter 2 immediately provide ideas for the empirical work. As the starting chapter for the empirical study part of the dissertation, chapter 3 introduces the pricing tool for Heston stochastic volatility model, the Finite Difference Method. Comparisons are made between my results and results from other authors. With this effective pricing tool, I then examine the sensitivity of theoretical option prices with respect to various model parameters. I found that it is those parameters involved in the equation of risk aversion that matter most in terms of option pricing. This consistency hinted me that when we calibrate the Heston model to the market, all price differentials should be mainly reflected in the market risk aversion. This is exactly the intuition for extracting the market risk aversion across strikes. To realize this idea, I decide to use cross-sectional fitting to extract relevant parameters, then apply the theoretical relation between risk premium and risk aversion in chapter 2 to calculate the market risk aversion. Two models are considered. One is the unrestricted model where all model parameters are allowed to change; the other is restricted model where only volatility risk is allowed to fluctuate and others are

estimated through Efficient Method of Moments (EMM). Hence in chapter 4 I introduce the EMM algorithm and present the estimates for various model parameters. Equipped with these computer algorithms, I in chapter 5 present the final results on implied market risk aversion from S&P 500 index options. I find an implied risk aversion smile, in both unrestricted and restricted models, across the strikes. This suggests that investors trading different options exhibit different attitudes toward risk and those who are extremely risk averse or risk loving tend to trade deep in or out of the money options. Another observation is that the volatility premium is very small compared with the asset risk premium, suggesting that the major component of market risk aversion comes from the Black-Scholes risk aversion, which also corresponds to the at the money option implied risk aversions.

While I was waiting for the numerical results, Professor Turvey asked me about the implied risk aversion behaviors with respect to the time to maturity of options. I responded that I do not have the data. He then suggested to look back at the results I got a year ago using S&P 500 index futures options data. So at the end of chapter 5 I presented those results which are very interesting by themselves.

Lastly in chapter 6 I summarized what I find both in theory and in reality. Future research is also identified.

CHAPTER 1

INTRODUCTION

That no arbitrage implies the existence of an economy-wide pricing kernel (or stochastic discount factor) is now a well-known result. Since it is proved in Rubinstein (1976), Garman (1976), Ross (1978), Hansen and Richard (1987), the whole asset pricing literature seems to take this pricing kernel approach more as a framework than a specific model. Both the consumption-based general equilibrium approach and the later risk-neutral approach are best regarded as different specifications of the pricing kernel. As Cochrane (2000) stated, “*All* asset pricing models amount to different functions for m (the pricing kernel)”. However, it is not until recently (Jackwerth (2000), Ait-Sahalia and Lo (2000), Rosenberg and Engel (2002), Bliss and Panigirtzoglou (2002)) that researchers start to estimate the empirical pricing kernel (EPK). In comparison, a closely related concept implied state price density (iSPD) has long been studied since Breeden and Litzenberger (1978) finds that the SPD can be recovered from the second derivative of option prices with respect to the strike prices. Jackwerth (2004) provides a complete survey on this respect. Since the pricing kernel is the ratio of the SPD and the objective probability density, Jackwerth (2000) derived the EPK based on a simultaneous estimation of the risk neutral density and the actual density. That the EPK he derived is locally increasing against wealth levels is termed the pricing kernel puzzle. Although a few dissertations have explicitly dealt with the puzzle (Branger and Schlag (2002), Ziegler (2007), and Brown and Jackwerth (2003)) no definitive answer has been provided so far.

Empirical estimates of market risk aversion (MRA), too, have received considerable research interests. Equity premium literature, examples including Friend and Blume (1975), Kydland and Prescott (1982), Mehra and Prescott (1985), estimate the MRA using consumption and equity return data. Bartunek and Chowdhury (1997) is the first to estimate the MRA from option prices. Assuming a power utility function (hence the form of the pricing kernel) they estimate the coefficient of constant relative risk aversion (CRRA). Adding exponential utility functions Bliss and Panigirtzoglou (2004) obtain implied MRA using British FTSE 100 and S&P 500 index options. Kang and Kim (2006) extend the analysis by assuming wider classes including HARA, log plus power and linear plus exponential utility functions. Benth, Groth and Lindberg (2010) estimate the implied MRA from a utility indifference approach using an exponential utility assumption. They find a smiling implied MRA across the strikes and maturities under stochastic volatility model, indicating certain crash risk not captured by the model. Along similar lines, Balckburn (2008) extracts implied MRA and inter-temporal substitution assuming an Epstein-Zin utility function. He finds reasonable estimates of the risk aversion parameter and the importance of differentiating the risk aversion and inter-temporal substitution. Perhaps more importantly, he claims that changes to the two concepts are both closely related to changes of the market risk premium. On the other hand, the relation between MRA and the pricing kernel (MRA is the negative of the derivative of log pricing kernel) is now exploited to non-parametrically estimate MRA from EPK. And because of this relationship, the pricing kernel puzzle is equivalent to observing negative empirical risk aversion. Perignon and Villa (2001) examine the EPK and iMRA from the French derivatives market. Jackwerth (2004) provides a complete survey on this regard. In general, iMRAs obtained in options market are substantially lower than those obtained in the consumption-based equilibrium literature.

By sharp contrast, little research has been done on another critical concept that has coexisted with the history of asset pricing. The concept of market price of risk (MPR) first appears in the traditional CAPM theory as the excess market portfolio return in units of the portfolio risk. The same concept shows in the risk-neutral pricing literature. As a parameter in the Radon-Nikodym derivative process, it determines the possible set of risk-neutral probability measures (Shreve (2003)). In a complete market model martingale restriction (or equivalently the existence of risk-neutral probability measures) turns out to be equivalent to the fact that the MPR equals the asset's instantaneous Sharpe ratio. The concept is omnipresent in incomplete market models. It was suggested in Bollen (1997) that “when pricing non-traded assets...the solution...depends on the assumed specification for the price of risk”. We see the market price of volatility risk in stochastic volatility models; in the term structure models we have the market price of interest rate risk; in oil contingent claim pricing models we have market price of convenience yield risk. In fact, to the randomness of each state variable there corresponds a market price of risk for this particular state variable. It is quite possible that the market price of risks for different state variables behave differently. The form of the MPR is also sometimes critical in asset pricing. Bollen (1997) demonstrates that “an incorrect specification (of the market price of risk) can have dramatic consequences for derivative valuation”. Cheridito, Filipovic and Kimmel (2005) shows that yields from an extended specification of market price of interest rate risk fits US data has better time-series behavior than those from standard specifications.

Given the importance of the concept, it is quite surprising to find very little literature devoted to it until very recently. Even among those few researches that actually touch upon the topic, a majority of them are on the market price of interest rate risk which is

usually not the central topic but rather as an unavoidable parameter whose form is then quickly assumed. Examples include Vasicek (1977), Hull and White (1990), Cox, Ingersoll and Ross (1985). Brennan and Schwartz (1979) introduce two MPRs-the market price of instantaneous interest rate risk and the market price of long term interest rate risk-to a term structure model. For convenience they assume the latter MPR to be a constant and use bond prices to extract the implied market price of instantaneous interest rate risk. By comparison, other kinds of market price of risks are largely neglected except the following dissertations. Gibson and Schwartz (1990) test a model with the spot oil price and the net convenience yield as two state variables. Using weekly oil futures contract prices they estimated the implied market price of convenience yield risk. Weron (2008) extracted empirical time-series market price of risk from Asian-style electricity options written on the spot electricity price traded at Nord Pool. Using a jump-diffusion model where the diffusion part follows a Vasicek mean-reversion process he found that the assumption of a constant market price of risk does not hold against real data. Instead the implied MPR “changes sign and varies significantly during the study period”. An interesting phenomenon he observed is that the time-series behavior of MPR seems to have close linear relationship with that of futures prices. Liu, Longstaff and Mandell (2006) estimates parameters of a five-factor affine term structure model using maximum likelihood and derives market price for liquidity and default risks inherent in interest rate swaps. They found that the liquidity premium varies with time significantly and has negative values in the 1990s. Turvey (2006) is another pioneer to derive empirical MPR. Using option prices on live cattle futures, he found an implied MPR smile for put options and frown for call options. Again the implied MPR changes sign and varies with time significantly. The study also suggests that the implied volatility smile problem is the same as the implied MPR smile/frown problem. All dissertations above reveal that the MPRs are very time-

varying and possibly follow their own stochastic processes. Gibson and Schwartz (1990) shows that the model performs much better after relaxing the assumption that the MPR is stationary. Liu, Longstaff and Mandell (2006), and Ahmad and Wilmott (2007) provide the time-series MPR which strongly support this conjecture. The latter even proposes a stochastic market price of interest rate risk model and hence introduces an “odd” concept-the market price of market price of risk risk. It is also suggested that further work needs to be done on the “functional relationship between (the MPR) and one or both state variable(s)... to enhance the performance of the model” (Gibson and Schwartz (1990)). As a byproduct of their dissertation, time series of market price of asset risk and market price of volatility risk are presented in Chernov and Ghysels (2000). They estimate both risk-neutral and subjective model parameters from the Heston model using efficient method of moments (EMM) on daily S&P 500 index contracts. Market prices of risks are calculated as the links between these parameters. Time series of the MPRs are driven by the hidden volatility process which is obtained by the re-projection method of the EMM algorithm. Turning to the energy market, Doran and Ronn (2008) find significant market volatility risk premium by combining implied and realized volatility in a two-step estimation procedure. Model parameters are estimated with GMM by taking advantage of a known relationship between instantaneous variance and expected variance under risk-neutral probability measure. Then the volatility risk premium is extracted by calibration to the real market options data. They also demonstrate by simulations that the market volatility risk premium is the crucial factor in explaining the implied volatility smile. Pirrong and Jermakyan (2008) use inverse methods to extract the market price of demand risk with data from a power market. They conclude that the market price of risk is quantitatively large and ignoring it may cause significant errors in valuing power-related contingent claims. In a more indirect way, Coval and

Shumway (2001), by examining expected option returns, find market price of volatility risk is priced into the option contracts and is an important factor in asset pricing. Bakshi and Kapadia (2003) also conclude after investigating the performance of delta-hedged option portfolios that a negative market volatility risk premium¹ is required to explain the non-zero (negative) portfolio gains.

Among these three fundamental concepts in asset pricing: market pricing kernel, market risk aversion and market price of risk, only the relationship between the pricing kernel and the market risk aversion has been widely studied. It is a common sense that the market risk aversion is closely related to the market price of risk, but until now there has been no explicit formula available under any model specifications. One important contribution of this dissertation is that we obtain a formula linking the MRA and the MPR under a general multifactor stochastic volatility model. This is done by revealing the close relationship between MPR and the market pricing kernel through a new definition of MPR in the spirit of Garman (1976). Defining the MPRs as proportional to the instantaneous change of the pricing kernel with respect to the corresponding state variables, we interpret MPRs as measurements of the sensitivity of Arrow-Debreu security prices with respect to the instantaneous continuous shift of corresponding state variables. We show that under martingale restriction assumption this new definition of MPR yields the well-known Sharpe ratio type of market price of asset risk.

One big confusion about market price of volatility risk in the current literature is that most studies treat a correlation-weighted average of market price of asset risk and

¹ However, as will be pointed out later in the dissertation the market price of volatility risks in these papers are actually a correlation weighted average of asset risk and volatility risk. Hence their results of a large volatility risk premium are overestimated.

market price of volatility risk as the market price of volatility risk itself. For example, Bates (1996) equation (5), Coval and Shumway (2001) equation (10), Doran and Ronn (2008) equation (4) and many others. As a consequence of this confusion, the conclusions of existence of significant volatility risk premium are actually implying significant total risk premium. We show in this dissertation by numerical experiment that the market price of asset risk is in fact the most important factor in asset pricing and explaining the volatility smiles.

Very few dissertations have derived implied MRA cross-sectionally with respect to the strikes and time to maturities. The formula linking MPR and MRA derived in this dissertation provides a way to do so. Adopting Heston's model we first estimate the model parameters using EMM as proposed by Gallant and Tauchen (1996). Since it is now well known that the regular Euler scheme is not suitable for certain combinations of parameters, we use the QEM by Andersen (2006) instead to discretize the continuous processes of both the underlying and the hidden volatility. Once obtaining the parameter estimates we minimize the distance between model-predicted option prices and market observed data to extract the implied market price volatility risk. Implied MRA is then obtained by our formula. Our results show that the market price of volatility risk is quantitatively small; the large value of negative volatility risk premium reported in previous studies is due to the market price of asset risk component and the negative sign is due to the leverage effect. Once we separate clearly the two market prices of risks, the volatility risk premium becomes much smaller. More interestingly, the risk aversion exhibits smiling effect for call options and frowning for put options cross-sectionally. If Heston model is able to capture all dynamics of the underlying data generating process, we should expect to observe a flat iMRA line across the strikes. This leads to the conclusion that there may be other risk

factors that are priced into the contracts yet not captured by the model. This result is consistent with many previous findings that Heston model is not good enough. The absolute level of risk aversion is also commensurate with existing options literature that implied risk aversion from options market tends to be much smaller than those found in the equity market.

Another discovery of the dissertation is about the martingale restriction assumption commonly made in the option pricing literature. Turvey (2006) casts doubts on this assumption by finding smiling/frowning market price of asset risks for calls/puts under the Black-Scholes setting. If martingale restriction, or equivalently, risk neutral measure holds, then constant MPRs should be expected. However, since Black-Scholes model is now known to at best a starting benchmark model, the findings could be just a reflection of the deficiency of the model itself. This dissertation adopts more general stochastic volatility models which are supposed to be more robust than Black-Scholes. Yet we again find similar smiling/frowning effects on the market price of asset risk, which strengthens our doubt of the risk neutral paradigm.

In a summary, the concept of market price of risk is perhaps the most important but also confusing in modern finance. Its relation with the market risk aversion and market pricing kernel is far from clear. Few attempts have been made to explain the volatility smile from an economic theoretical point of view. Inspired by these deficiencies in the current literature, this dissertation aims to achieve the following objectives: first, theoretical relationships between the market pricing kernel, market prices of risks and market risk aversion will be established by considering a pure exchange economy with risk factors driven by a stochastic volatility model completed with put options. The equilibrium relation between the pricing kernel and the market prices of risks provides

additional insights of what the market prices of risks actually mean, while the equilibrium relation between market prices of risks and risk aversion lays a foundation for empirically extracting the market risk aversion from real market data; second, empirical risk aversion will be extracted using S&P500 index options data across the strikes. The hypothesis is that the empirical risk aversion will exhibit certain smiling or frowning effect across the strikes, corresponding to the observed implied volatility curve. In order to achieve this, a two-step method is planned. In the first step, model parameters will be estimated using EMM combined with QEM algorithm. In the second step, the squared difference between the theoretical value and the market price is minimized to extract the market price of volatility risk (the market asset risk is equal to the Sharpe ratio by risk neutrality assumption). For that purpose, a Finite Difference Method and a Monte Carlo simulation-based pricing method will be programmed using Excel/VBA. Third, to further understand the role played by market prices of risks in option pricing, the dissertation will run a series of computer simulations to examine the pricing effect due to all model parameters including the two market prices of risks.

The rest of the dissertation is organized as follows. Chapter 2 explores the equilibrium relations between the market prices of risks, risk aversion and market pricing kernel under a pure exchange economy where the traded assets are driven by a stochastic volatility model completed by liquidly traded options. The application to the special case of Black-Scholes and Heston stochastic volatility model are investigated. As a test of robustness of the results, habit formation preference is examined. Chapter 3 introduces finite difference methods to solve the pricing partial differential equation. Numerical experiments are conducted using the finite difference method to examine the relation between model parameters including the market prices of risks and the

volatility smile. It shows that it is the asset risk premium, not volatility risk premium that determines the shape of smiles. Chapter 4 describes the quadratic-exponential martingale correction method (QEM) used to discretize the continuous time diffusion model of Heston, introduces efficient method of moments (EMM) and presents the results of estimated model parameters. Chapter 5 calibrates the Heston model to the real S&P 500 index options data to extract the market price of volatility risk and market risk aversion across strikes. Chapter 6 concludes and identifies future research areas.

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CHAPTER 2

EQUILIBRIUM MARKET PRICES OF RISKS AND MARKET RISK AVERSION

Risk neutral pricing approach is today's main stream of options pricing theory. The core idea of this approach is based upon the fact that wealth levels in different states of the world have different marginal utility values to market players. This implies that the naive discounted expected value of asset payoffs based on the objective probability distribution will certainly produce biased asset prices. Instead, a *marginal utility weighted* risk neutral probability should be used in pricing equations. In the very special and rare case of a complete market where every asset can be replicated by other traded assets, theory predicts that the marginal utility of each market player will converge to the point that a unique pricing kernel of asset pricing comes to existence. More interestingly under this circumstance the market price of risk equals the Sharpe ratio of the underlying asset and does not enter the pricing equation. Under an incomplete market assumption, however, since there are some risk factors market players cannot hedge away, the market rewards those who are willing to bear risks; hence the market prices of these risks appear in the pricing equation inevitably. In a continuous stochastic time-state world where randomness are modeled by Brownian motions, using marginal utility weighted probability corresponds to constructing a Radon-Nikodym stochastic process that links the objective probability measure and the risk neutral probability measure. What is critical for our purpose is that the market price of risk process involved in the Radon-Nikodym process is the only term that captures the marginal utility flavor. Therefore at least intuitively a relation between the marginal utility, market price of risk and market risk aversion

should exist. As stated in Bakshi and Kapadia (2003), “In general, the volatility risk premium will be related to risk aversion and to the factors driving the pricing kernel process”.

This chapter proceeds like this: Section One identifies the contributions and review the literature. In Section Two, we set up the model by considering a pure exchange economy driven by stochastic volatility model with habit formation preferences. Section Three examines the equilibrium relationship between the market price of risk, risk aversion and market pricing kernel. Formulas under both single-state-variable and multi-state-variable models are given respectively and examples under each circumstance are discussed. As an independent part, Section Four looks at option pricing using a market price of risk approach. Section Five concludes.

Section I - Introduction

1.1. Contributions

This dissertation considers the equilibrium relationships between the market pricing kernel, the market prices of risks (MPRs) and the market risk aversion (MRA) under a pure exchange economy where utility exhibits habit formation. The state factors are modeled by a continuous time stochastic volatility model completed by liquidly traded options. Our contribution is in four aspects. *First*, we extend the economic analysis of market risk aversion in Jackwerth (2000) by incorporating stochastic volatility and habit formation utility to the model. We offer a possible solution – existence of habit formation utility for the representative agent – to the empirical pricing kernel puzzle first documented by Jackwerth (2000). Specifically, if the future disutility induced by a rising future standard of living is greater than the increase in contemporaneous utility

then it is at least possible in theory that the market risk aversion exhibits local negativity. *Second*, we derive the quantitative relationship between market prices of risks and market risk aversion under stochastic volatility and habit formation. To our best knowledge this has not been done in the current literature. Although it is common sense that the market risk aversion is closely related to the market price of risk, until now there has been no explicit formula available under multiple state factor models. More interestingly, we show that under certain assumptions on the economy and time separable preferences market risk aversion is a linear combination of the market prices of risks. This relationship can be utilized to extract empirical market risk aversion from implied market prices of risks, which circumvents the debatable approach of using market implied risk neutral density and subjective density function as in many dissertations (Jackwerth (2000), Ait-Sahalia and Lo (2000), Rosenberg and Engel (2002), Bliss and Panigirtzoglou (2002), Perignon and Villa (2002)). *Third*, we verify through the equilibrium conditions imposed on the pricing kernel and market prices of risks that the risk neutral pricing partial differential equation is a restricted version of the fundamental pricing equation in Garman (1976), hence providing a potential test for risk neutrality assumption commonly made in the modern option pricing literature. *Last*, one byproduct of the dissertation is a better interpretation of the market prices of risks under diffusion models. In conventional asset pricing literature the market price of risk is loosely defined to be “the return in excess of the risk-free rate that the market wants as compensation for taking risk”. In comparison, in the diffusion models literature market price of risks are parameters in the Radon-Nikodym derivative process whose value determines the possible set of risk-neutral probability measures. Depending on whether one can or cannot hedge away the risk using any or combined currently traded assets in the market, the market price of risk may be present or absent in final pricing equations. It turns out that in the simplest

case where the asset risk can be perfectly hedged with the underlying asset the market price of risk is exactly the Sharpe ratio of the underlying, consistent with the classical asset pricing literature. However, this definition becomes very hard to interpret in incomplete market models where we face awkward concepts such as the market price of volatility risk, the market price of interest rate risk and the market price of convenience yield risk. Since the concept is defined through the pricing kernel in the diffusion models literature, it is natural to interpret the concept of MPR through the pricing kernel. In general, the Radon-Nikodym process is defined as:

$$\xi(t) = \exp \left\{ \sum_{i=1}^m \left(-\frac{1}{2} \int_0^t \lambda_{iu}^2 du - \int_0^t \lambda_{iu} dB_{iu} \right) \right\}$$

where λ_i is the MPR corresponding to the i -th risk factor, B_i is a Brownian motion under the subjective probability measure, m the number of risk factors. This implies that ² $\sum_{j=1}^m \lambda_{it} \rho_{ij} = -\text{cov}(\frac{d\xi_t}{\xi_t}, \frac{dS_{it}}{S_{it}}) / \sigma_{it}$, where ρ_{ij} is the correlation of Brownian motion B_i and B_j . In other words, the market price of the i -th variable risk, together with the market prices of risks for other state variables through their correlations, describes the correlation between the percentage change of the pricing kernel and the percentage change of the i -th variable. This relation has been mentioned in Detemple and Zapatero (1991), Coval and Shumway(2001) and Bakshi and Kapadia(2003). We argue that our derived equilibrium relation between market prices of risks and market pricing kernel, $\bar{\lambda}_t = -D_t^T \nabla' G_t / G_t$, is more economically meaningful than the traditional definition in interpreting the nature of market price of risk. This makes intuitive sense because the MPR is after all the price agreed on by market players for the corresponding risk factor when the market reaches equilibrium. As an example, the

² A related note is that the quadratic variation (QV) of the log pricing kernel process is determined fully by the MPRs and the correlations between randomness involved. In a simple model of a single random source, the QV is only determined by the MPR for that state variable.

formula implies that in equilibrium the market price of volatility risk is proportional to the percentage change of Arrow-Debreu security prices with respect to an instantaneous change of volatility resulting from its nature of riskiness on some future date. Intuitively the MPR captures how the market players perceive about the riskiness embodied in the corresponding risk factor in units of utility.

1.2. Literature Review

Empirical estimates of market risk aversion (MRA) have received considerable research interests. Equity premium literature (Friend and Blume (1975), Kydland and Prescott (1982), Mehra and Prescott (1985) and many others) estimates the MRA using consumption and equity return data. Efforts are also devoted to estimating MRA from derivatives market. Sprenkel (1967) extracted MRA using his warrant pricing formula involving risk aversion parameter. Bartunek and Chowdhury (1997) is the first to estimate MRA from option prices. Assuming a power utility function (hence the form of the pricing kernel) they estimate the coefficient of constant relative risk aversion (CRRA). Adding exponential utility functions Bliss and Panigirtzoglou (2004) obtain implied MRA using British FTSE 100 and S&P 500 index options. Kang and Kim (2006) extend the analysis by assuming wider classes including HARA, log plus power and linear plus exponential utility functions. Despite the facility of implementing the models, however, results derived using this preference-based approach may be misleading because: (i) The models employed by above fail to incorporate either stochastic volatility which should be included for any reasonable option pricing model, or non-time-separable utility functions; (ii) As demonstrated in He and Leland (1993), assumptions on particular forms of utility functions inevitably impose strong restrictions on equilibrium diffusion coefficients of the state factor processes. In an attempt to dealing with the former issue, Benth, Groth and Lindberg

(2010) estimate the implied MRA from a utility indifference approach using an exponential utility assumption under stochastic volatility model. They find a smiling implied MRA across the strikes and maturities, indicating certain crash risk not captured by the model. In another attempt, Balckburn (2008) extracts implied MRA and inter-temporal substitution assuming an Epstein-Zin utility function. He finds reasonable estimates of the risk aversion parameter and the importance of differentiating the risk aversion and inter-temporal substitution. Perhaps more importantly, he claims that changes to the two concepts are both closely related to changes of the market risk premium, which is consistent with what we derived. On the other strand of research, the relation between MRA and the empirical pricing kernel (EPK) (MRA is the negative of the derivative of log pricing kernel) is exploited to estimate MRA from non-parametric EPK. Considering a pure exchange economy with time-separable preference, Jackwerth (2000) derived the EPK based on a simultaneous estimation of the risk neutral density and the subjective density. That the EPK he derived is locally increasing against wealth levels (or equivalently the market risk aversion is locally negative) is termed the pricing kernel puzzle. Using different estimation methods for the EPK researchers have observed different levels of MRAs and shapes of EPKs.³ However, Singleton (2006) doubt that these findings may not be robust due to the simplified assumptions made about the underlying economy and the dimensionality of the state vector. Approaches to explaining the puzzle include Ziegler (2007), and Brown and Jackwerth (2004). No definitive answers are provided so far.

In this dissertation we propose to a new method to estimate MRA through the equilibrium relation between MRA and MPRs. More generally, triangular equilibrium relations between the market pricing kernel, market prices of risks and the market risk

³ Jackwerth (2004) provides a complete survey on this regard.

aversion under a pure exchange economy are derived. To justify the existence of a representative agent we complete the market with liquidly traded put options. Since each put option corresponding to a specific strike level and maturity date completes the market, the model implies a “sequentially complete” market as options expire. However, for a fixed maturity date, it is reasonable to expect that the representative agent exhibits constant risk aversion across the strikes. It leaves an empirical test to see if this is true. Equilibrium relation between the MPRs or risk premium and the pricing kernel is examined in previous studies. He and Leland (1993) consider a pure exchange economy endowed with one unit of risky asset. Consumption happens only at the final date and the representative agent’s preference is described by a time-separable von Neumann-Morgenstern utility function. They find that the equilibrium condition restricts the asset price process coefficients to satisfy a partial differential equation. In particular, a quantitative relation between the market price of risk and the pricing kernel is obtained at the terminal date.⁴ Pham and Touzi (1996) extend the analysis of He and Leland (1993) by introducing stochastic volatility factor to the model and allowing intermediate consumption. Similar quantitative relationship between the market risk premium (hence the market prices of risks) and the market pricing kernel is obtained for all time t . A generalized system of partial differential equations for “viable” risk premiums is derived.⁵ Following the analysis in He and Leland (1993) and Pham and Touzi (1996), we conclude that the inclusion of stochastic volatility does not affect the equilibrium relation between the market risk premium and the pricing kernel as long as the market can be completed by traded relevant contingent claims. It is worthy to mention that Pham and Touzi (1996) obtained similar results for market prices of risks. Using a martingale approach, they

⁴ See equation (8) in Theorem 1 in their paper.

⁵ See Theorem 4.1 in their paper.

established necessary and sufficient conditions for an equilibrium “viable” risk premium. In contrast, we adopt the dynamic programming approach to derive the relationship between market price of risk and the pricing kernel. We also show that in equilibrium the pricing kernel can be supported by some utility functions and is path-independent for von-Neumann Morgenstern preferences. Taking advantage of the market completeness we then turn to the static version of the investor optimization problem and derive the equilibrium relationship between the market pricing kernel and the market risk aversion. Combining the two equilibrium conditions, we establish the equilibrium relation between the MPRs and the MRA. This allows one to empirically extract the MRA without estimating the risk neutral density and the subjective density.

To test the robustness of our result, we consider a habit formation utility besides time-separable utility functions. Habit formation utility has been successfully applied to explain the equity premium puzzle. Optimal consumption and portfolio choices under habit formation have been extensively studied in the literature (Sundaresan (1989), Constantinides (1990), Detemple and Zapatero (1991), (1992), Campbell and Chocrane (1999), Detemple and Karatzas (2003), Egglezos and Karatzas (2007) etc). Detemple and Zapatero (1991) provides a first order condition for the representative agent’s optimization problem,⁶ showing that in optimum the state price is related not only to the contemporaneous marginal utility, but also to the expectation of all future disutilities through the future increasing living standard. This effectively makes the pricing kernel path-dependent. One important characteristic of the economy considered in their article is that the asset prices are endogenously determined by marginal utilities and parameters of an endowment process.⁷ Chapman (1998)

⁶ See equation (4.2) in their paper.

⁷ See equations (5.2) and (5.3) in their paper.

provides an example of endowment process that can lead to negative state prices when calibrated to the real world economic data. In our dissertation, we avoid this possibility by considering an economy with endogenous endowment. We find that the relationship between market price of risk and the market pricing kernel still holds even under habit formation in which case the pricing kernel is now path-dependent through an additional term involving living standard. However, the relation between MRA and market pricing kernel is different from that in the time-separable preference case. Future utilities induced from future living standards come into play, making MRA to be possibly instantaneously negative. If the future disutilities from increasing future living standard are greater than the marginal utility gained from contemporaneous consumption, the market representative agent will exhibit risk loving. In other words, if the marginal utility, after netting the future disutilities due to increasing living standard, becomes negative then the agent will be willing to pay more to take the risk. In this case, the risk aversion is no longer a linear function of the market prices of risks but with extra term reflecting the habit level.

The rest of the dissertation is organized as follows. Section II describes the economy and the model assumptions we make. Section III derives the equilibrium results about market prices of risks using a dynamic programming approach. We derive the first order condition for the corresponding static optimization problem and find the equilibrium market risk aversion. We then establish the relationship between the market prices of risks and the risk aversion. Applications to special cases such as Black-Scholes and Heston stochastic volatility models are investigated. Section IV concludes.

Section II –Model

Consider a complete probability space (Ω, \mathbb{F}, P) where Ω is the event space with a typical element ω , \mathbb{F} the sigma algebra of observable events, P a probability measure assigned on (Ω, \mathbb{F}) . Let I_t denote the information set faced by investors at time t , then the payoff space $\Gamma_t^+ = \{\Pi_t \in I_t : E[\Pi_t^2 | I_{t-1}] < \infty\}$ where Π_t is the asset payoff at time t . That is, the payoff space is the set of all random variables with finite conditional second moments given the previous period information. In this dissertation we focus on European options market where risky assets include the underlying asset and various options written on it. We use Γ_t to denote the payoff space on this market at time t , so $\Gamma_t \subset \Gamma_t^+$ for all time t .

Assumption 2.0: *The market is efficient such that the law of one price (value-additivity) and no arbitrage principle hold.*

The former implies the existence of a pricing kernel $G(t)$ with respect to its corresponding payoff space Γ_t under loose regularity conditions (Hansen and Richard (1987), Cochrane (2001)), while the latter, combined with complete market assumption, indicates a unique and strictly positive pricing kernel on the payoff space Γ_t . This allows us to write a pricing function

$$(0.1) \quad p_t = E[G(t, T)\Pi_T | I_t] \text{ for all } \Pi_T \in \Gamma_T$$

where p_t is the risky asset price at t . The expectation is taken under the original probability measure P . The key point here is that the future payoff needs to be adjusted

before taking expectation. Under the classical asset pricing literature, the weight is the representative investor's marginal utility based on the idea that different levels of wealth have different marginal utility meanings. In states where the individual is already wealthy she values the cash flow less than in those states where she is relatively poor. Specifically, the classical asset pricing models assume a complete market where a representative agent with regular von-Neumann Morgenstern preference exists. First order conditions of the investor's optimization problem as formulated in Merton (1973) say that the pricing kernel $G(t)$ is then equivalent to the normalized inter-temporal marginal rates of substitution (Lucas (1978), Rubinstein (1976)): $G(t, S_t) = \phi_t \frac{U'(S_t)}{U'(S_0)}$ where ϕ is the *subjective* stochastic discount factor that captures the impatience of the representative investor. Here the state variable is assumed to be the aggregate wealth level.⁸ Another way to weight the terminal payoff before taking expectation is to change the probability structure under original probability measure P to that under a new probability measure Q . Since the new measure is constructed such that all discounted payoffs are martingales, this approach is called risk neutral approach, which allows us to rewrite (2.1) as follows:

$$(0.2) \quad p_t = E^Q[e^{-r(T-t)} \Pi_T | I_t]$$

Q is the risk-neutral probability measure defined as $\frac{dQ}{dP} = \xi(T)$ where

$$\xi(\lambda, t) = \exp \left\{ \sum_{i=1}^m \left(-\frac{1}{2} \int_0^t \lambda_{iu}^2 du - \int_0^t \lambda_{iu} dB_{iu} \right) \right\}$$

⁸ In most empirical applications the state variable is chosen to be the wealth portfolio with equity index level as its proxy. Using iterated conditional expectation, Rosenberg and Engel (2002) and Cochrane (2000) claim that under loose assumptions a pricing kernel projected onto the equity index level is the same as the original pricing kernel. Even without those assumptions the projected kernel shares the same interpretation as the original one. Therefore in this paper consumption is replaced by equity index level and the pricing kernel G is the projected kernel onto the equity index space.

The pricing kernel $G(T) = e^{-rT} Z(T)$. Unlike the previous preference-based approach where one has to deal with unobservable utility functions, the risk neutral approach writes the pricing kernel as a stochastic process characterized by market prices of risks corresponding to each random source. In some cases, the two approaches are identical. For example, in the classical Black-Scholes world with lognormal asset price process, the model implicitly assumes a pricing kernel that is in the power utility form. In most cases, however, the pricing kernel written as a reduced form right from the beginning as in the no-arbitrage approach are not always associated with any investor's marginal rate of substitution even when the market has a representative agent (Garman (1976), Singleton (2006) chapter 8). Moreover, He and Leland (1993), Pham and Touzi (1996) show that when the pricing kernel can be supported by some utility function the market prices of risks must satisfy certain partial differential equations to be consistent with market equilibrium. Therefore, we should proceed very carefully in any attempt to derive the relationship between market risk aversion and pricing kernel under continuous time diffusion-based model.

2.1. The Risky Factor Processes

Assumption 2.1: *The risky factor processes form a single-factor stochastic volatility model*

Classical Black-Scholes model is a complete market model as all options can be perfectly replicated by certain number of shares of stocks and bonds. However there are well known deficiencies in the model as capitalized in implied volatility smiles. One most challenged assumption is the log-normality of the underlying asset

distribution. Various stochastic volatility models⁹ are henceforth proposed to capture skewness and excess kurtosis of the stock distribution to replicate the observed smiles in the options market. In this dissertation we focus on a single-factor stochastic volatility model for two reasons. First, as will be discussed in Assumption 2.2 recent research results show that under very general conditions the single-factor model can be completed by adding a liquidly traded contingent claim such as a put option. The completeness provides a theoretical convenience of guaranteed existence of a representative agent for the economy. Second, we could have assumed homogeneous preferences to avoid the aggregation problem while keeping the market model incomplete. It turns out that in this case the equilibrium relationship between the market prices of risks and the market pricing kernel is the same in essence as in the single-factor model.¹⁰ The equilibrium market risk aversion equation, however, does not hold anymore since the stochastic control problem faced by the representative agent does not necessarily translate into a static optimization problem due to the market incompleteness. More specifically, a single-factor stochastic volatility model is as follows:

$$\begin{aligned}
(0.3) \quad & dS_t = [\mu_1(t, S_t, \varsigma_t) - D(t, S_t, \varsigma_t)]dt + \sigma_1(t, S_t, \varsigma_t)dW_{1t} \\
& d\varsigma_t = \mu_2(t, S_t, \varsigma_t)dt + \sigma_2(t, S_t, \varsigma_t)dZ_t \\
& dZ_t = \rho_t dW_{1t} + \sqrt{1 - \rho_t^2} dW_{2t}
\end{aligned}$$

where (W_{1t}, W_{2t}) is a multi-dimensional Brownian motion defined on (Ω, \mathbb{F}, P) . Let the first risky factor be S_t , the underlying asset process, the dividend process

⁹ See, for example, Chernov & Ghysels (2000) for descriptions of a two volatility factor stochastic volatility model and a stochastic volatility model with jump process.

¹⁰ This shows again the robustness of the equilibrium relation between the market prices of risks and the market pricing kernel.

$D_t = \delta_t S_t$ with dividend rate δ_t , the second risky factor $\varsigma_t = \sigma_{1t}^2$ the stochastic volatility, ρ_t the instantaneous correlation between the two Brownian motions. Many well known stochastic volatility models such as Hull-White, Heston, SABR are specializations of (3.1). We can write (3.1) in matrix form as

$$d\bar{S} = \bar{\mu}dt + D \bullet d\bar{W}_t \text{ where } \bar{S}_t = \begin{pmatrix} S_t \\ \varsigma_t \end{pmatrix} D(t, \bar{S}) = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{pmatrix}$$

$$\bar{\mu}(t, \bar{S}) = \begin{pmatrix} \mu_1(t, \bar{S}) - D(t, \bar{S}) \\ \mu_2(t, \bar{S}) \end{pmatrix} \text{ the instantaneous drift vector. Let}$$

$$\Sigma(t, \bar{S}) = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \text{ be the instantaneous covariance matrix, then } DD^T = \Sigma. \text{ We}$$

assume Σ is full rank hence invertible.

Assumption 2.2: *Model (3.1) can be completed by traded contingent claims.*

One problem embodied in stochastic volatility models is that they introduce market incompleteness through the newly-added hidden volatility process which cannot be hedged by trading stocks and bonds alone. This market incompleteness creates serious issues in interpreting the market risk aversion as the existence of a representative agent is not guaranteed. Recent studies, however, have shown that if we add in liquidly traded options into the model, then under some conditions the model becomes complete. Romano and Rouzi (1997) shows that under restrictive conditions, i.e. model coefficients are independent of the underlying asset price, any European contingent claims complete the market. Davis and Obloj (2007) extend conditions proposed in Davis (2004) to a necessary and sufficient condition for market completeness using vanilla options. They prove that if the pricing function is real

analytic and the gradient matrix of pricing function with respect to risk factors is locally invertible at some point, then the market is complete.¹¹ The pricing function being real analytic is shown to be easy to satisfy as put options have bounded payoff and pricing function for calls can be derived from put-call parity. In a slightly different approach where asset prices, instead of risk factors, processes are modeled, Jacod and Protter (2007) show that an unstable complete market may be obtained by imposing some complex compatibility conditions between underlying asset price and option prices. Carr and Sun (2007) suggest that under certain hypothesis the market can be completed by variance swaps now liquidly traded. In this dissertation we adopt the approach as in Davis (2004) and Davis and Obloj (2007). That is, we complete the market with appropriate put options, model the risk factor processes directly and take the asset prices as conditional expectations.

In particular, we assume the market has one unit supply of the underlying asset S_t and zero net supply of a riskless asset with an instantaneous interest rate r_t . There are European put options $\{V_{ij}\}$ with $i=1, \dots, M$, $j=1, \dots, N_i$ with time to maturities $0 \leq T_1 \leq \dots \leq T_i \leq \dots \leq T_M = T$, N_i the number of strikes for the contract with maturity T_i . According to Davis and Obloj (2007) the market is then completed by any of these put options. Specifically, if we are looking at the time interval of $[0, T_i]$ we can essentially use any put option with equal or longer time to maturity to complete the market. For each of these completed markets there corresponds a representative agent with a certain degree of risk aversion. Theoretically these risk aversions should not be significantly different across the strikes and time to maturities. Violations can be due to the model misspecification, market inefficiency. It will be an empirical exercise to see how implied risk aversion behaves across the strikes and time to maturities.

¹¹ See Theorem 4.1 in the article.

The corresponding pricing kernel process $G(t)$ is the discounted Radon-Nikodym process:

$$G(t) = \exp \left\{ -\int_0^t r_u du - \frac{1}{2} \int_0^t (\lambda_{1u}^2 + \lambda_{2u}^2) du - \int_0^t \lambda_{1u} dW_{1u} - \int_0^t \lambda_{2u} dW_{2u} \right\}$$

where λ_{1u} and λ_{2u} are adapted process of market price of asset risk and market price of volatility risk respectively. Define two new Brownian motions as:

$$\tilde{W}_{1t} = W_{1t} + \int_0^t \lambda_{1u} du, \quad \tilde{W}_{2t} = W_{2t} + \int_0^t \lambda_{2u} du$$

Using the fact that the combined stock and volatility process $\{S_t, \varsigma_t\}$ is Markovian, the

put option price can be written as

$$V(t, S_t, \varsigma_t) = E^Q \left[e^{-\int_t^T r_u du} (K - S_T)^+ | F_t \right]$$

Applying Ito's lemma, we have:

$$(0.4) \quad dV_t = \mu_3(t, V_t) dt + V_s \sigma_{1t} dW_{1t} + V_\varsigma \sigma_{2t} dW_{2t}$$

Equivalently we can write it as

$$(0.5) \quad dV_t = \mu_3(t, V_t) dt + F(t, s, \varsigma) d\hat{W}_t$$

where $F(t, s, \varsigma) = \sqrt{V_s^2 \sigma_1^2 + V_\varsigma^2 \sigma_2^2 + 2\rho V_s V_\varsigma \sigma_1 \sigma_2}$ and \hat{W}_t is a Brownian motion under risk-neutral measure with $dW_{1t} d\hat{W}_t = \rho' dt$ where $\rho' = \frac{\rho V_\varsigma \sigma_2 + V_s \sigma_1}{F}$.

2.2. A Representative Agent Economy

The economy is composed of investors who are utility maximizers over a finite time horizon $[0, T]$. We assume that the representative agent has initial endowment of one unit of the underlying asset S_0 and no exogenous endowments during the period. Let α_t denote the amount invested in the underlying asset and β_t the amount invested in the put option (both satisfying the usual integrability conditions) and e_t her wealth at time t . The agent's consumption is described by (c_t, e_t) with consumption rate process $\{c_t\}$ and final wealth e_T . The agent's wealth process is then described through her portfolio process $(\alpha_t, \beta_t, e_t - \alpha_t - \beta_t)$.

$$(0.6) \quad de_t = \left[\alpha \frac{\mu_1}{S} + \beta \frac{\mu_3}{V} + (e - \alpha - \beta)r - c \right] dt + \alpha \frac{\sigma_1}{S} dW_1 + \beta \frac{F}{V} d\hat{W}$$

Assumption 2.3: *The representative agent's preference exhibits habit formation as described by:*

$$E \left[\int_0^T e^{-\int_0^t \rho_s ds} u(c_t, x_t) dt \right], \text{ where } x_t = x_0 e^{-k_1 t} + k_2 \int_0^t e^{-k_1(t-s)} c_s ds \text{ is the habit level.}^{12}$$

¹² Refer to Sundaresan(1989), Constantinides(1990), Detemple and Zapatero(1991), Chapman(1998), Campbell and Cochrane(1999), Detemple and Karatzas(2002) for various versions of the habit level and the utility form.

Habit level x_t is an exponentially weighted average of past consumptions hence making the utility function path dependent. Ito's Lemma shows that the habit level x_t follows a differential equation $dx_t = (k_2 c_t - k_1 x_t)dt, x_0 \geq 0$. Assuming strict positive marginal utility at time 0 we consider an *additive* habit formation preference, i.e. the agent is forced to consume more than he does in the past. The utility function is increasing and strictly concave in c , strict decreasing in x and concave in (c, x) .¹³ Under this structure, an increase in consumption at date t decreases contemporaneous marginal utility but increases the marginal utility of consumption at future dates. A consumption and portfolio strategy (c_t, α_t, β_t) is *feasible* if it satisfies (3.4) with a nonnegative wealth process. We let $\Psi(S_0)$ denote the set of all feasible consumption-portfolio strategies with initial wealth S_0 .

The representative agent's investment problem is thus:

$$(0.7) \quad \max_{(\alpha_t, \beta_t, c_t) \in \Psi(S_0)} E \left[\int_0^T u(t, c_t, x_t) dt + U(e_T) \right]$$

The agent maximizes her expected utility from intermediate consumption and terminal wealth over all feasible consumption-portfolio strategies. Note that (3.5) is a stochastic control problem which can be solved by the well-known Hamilton-Jacobi-Bellman equation.

¹³ See Detemple and Zapatero (1991) for a full characterization of the utility function.

Section III - Equilibrium Conditions

We say that the market is in equilibrium if the representative agent optimally chose to hold the market portfolio – the one unit supply of risky underlying asset and zero unit of option, and she consumes all the output, in this case the dividends paid by the underlying asset. That is, the market clears for all t :

$$\alpha_t^* = S_t, \beta_t^* = 0, c_t^* = \delta_t \quad \forall t \in [0, T]$$

Hence in equilibrium the wealth is just the underlying asset $e_t^* = S_t \quad \forall t \in [0, T]$.

Following He and Leland (1993) we solve (3.5) using stochastic dynamic programming techniques.

3.1. Viable Market Prices of Risks

Proposition 3.1 summarizes the result and the proof is provided in the Appendix.

Proposition 3.1: *In an economy as specified in section II and asset price processes in (1.4) and (1.6), in equilibrium*

$$(1) \quad G(t) = \frac{J_e(t, S_t, S_t, \varsigma_t, x_t)}{J_e(0, S_0, S_0, \varsigma_0, x_0)} = \frac{u'(t, \delta(t, S_t), x_t) + k_2 J_x(t)}{u'(0, \delta(0, S_0), x_0) + k_2 J_x(0)}$$

$$G(T) = \frac{U_S(S_T)}{J_e(0, S_0, S_0, \varsigma_0, x_0)}$$

where $J(\cdot)$ is the value function of the optimization problem (3.5). i.e. $G(t)$ can be supported by some utility functions.

(2) if a pair of market prices of risks is viable, then: $\bar{\lambda}_t = -D_t^T \nabla' G_t / G_t$, ∇G_t is

the gradient of G with respect to the risk factors at time t , or equivalently

$\bar{\lambda}_t = -D_t^T \nabla_{T \rightarrow t}' G$ where $\nabla_{T \rightarrow t}' G$ is the partial derivative with respect to the risk

factors at time T then take the limit at time t , i.e.

$$(3) \quad \partial G_{s,T \rightarrow t}(t, S_t, \varsigma_t) = \lim_{\substack{S_T \rightarrow S_t \\ T \rightarrow t}} \left[\frac{\partial G(t, T; S_t, S_T; \varsigma_t, \varsigma_T)}{\partial S_T} \right],$$

$$\partial G_{\varsigma,T \rightarrow t}(t, S_t, \varsigma_t) = \lim_{\substack{\varsigma_T \rightarrow \varsigma_t \\ T \rightarrow t}} \left[\frac{\partial G(t, T; S_t, S_T; \varsigma_t, \varsigma_T)}{\partial \varsigma_T} \right].$$

(4) if a pair of market prices of risks is viable, then the pricing kernel $G(t)$ satisfy

the following partial differential equation (discounted Feynman-Kac):

$$\begin{aligned} G_t / G + (\mu_1 - \delta S) \partial G_s + \mu_2 \partial G_\varsigma + \partial G_x (k_2 \delta - k_1 x) + \frac{1}{2} \sigma_1^2 \partial G_{ss} + \frac{1}{2} \sigma_2^2 \partial G_{\varsigma\varsigma} \\ + \rho \sigma_1 \sigma_2 \partial G_{s\varsigma} + r = 0 \end{aligned}$$

where all derivatives are limit derivatives as defined above except the partial derivative with respect to time t .

Remark 1: although we consider a single-factor stochastic volatility model here, the proposition applies to any multifactor pure diffusion models. Condition (1) suggests that the pricing kernel in equilibrium can be supported by some utility function, hence can be interpreted as the representative agent's marginal rate of substitution. Although not shown explicitly, from the proof process we can see that this result in fact holds as long as the market is complete. Condition (2) suggests why λ is termed the *market price of risk*. Since the kernel function G is the present value of the Arrow-Debreu

security prices, G_s captures the (volatility scaled) sensitivity of current Arrow-Debreu security prices with respect to future fluctuations of the corresponding state variable. The market price of volatility risk is related to Arrow-Debreu security price change with respect to the volatility risk factor, while the market price of asset risk is affected by the Arrow-Debreu security price changes with respect to both the asset price factor and the volatility factor through their correlations. Condition (3) shows that all major derivatives of the pricing kernel with respect to its components are captured by the risk free rate in equilibrium. Hence any assumptions made on the risk free rate will impose restrictions on the pricing kernel and model coefficients. We can further differentiate (3) with respect to the two risk factors to derive the partial differential equations that viable market prices of risks must satisfy and show that they are sufficient conditions for viable market prices of risks. However, since our focus in this dissertation is on the equilibrium relations between the market prices of risks and risk aversion, and the major steps involved are quite similar to those in He and Leland (1993) and Pham and Touzi (1996),¹⁴ we will skip the proofs and assume instead that these sufficient conditions are satisfied. Note also that condition (3) is equivalent to saying that the pricing kernel correctly prices digital options.

Remark 2: in the time separable preference case, condition (1) is identical to He and Leland (1993) with intermediate consumption and Pham and Touzi (1996) equation (3.1), (3.2). The latter derived the condition using martingale approach. Here we follow He and Leland (1993) using dynamic programming approach. The extra term of the derivative of the value function with respect to the habit level is new. As the agent optimally chooses her consumption and portfolio strategy to maximize her utility with habit formation, she takes into consideration the past consumption which

¹⁴ See equation (7) in He and Leland (1993) and Theorem 4.1 in Pham and Touzi (1996).

in equilibrium is dependent on past underlying asset prices. This makes the market pricing kernel path-dependent. Condition (1) is another version of equation (4.1) in Detemple and Zapatero (1991) which expressed the pricing kernel as contemporaneous marginal utility net expected future disutilities induced from the habit level. The indirect value function in (1) essentially captures the same idea. Notice that although the pricing kernel is path-dependent, its equilibrium relation with market prices of risks as in (2) is the same as in the case of time-independent preferences. This is due to the fact that the prices of risks are only related to the volatility terms in the market pricing kernel, and the newly introduced habit level is assumed to be deterministic $dx_t = (k_2 c_t - k_1 x_t)dt$, hence only entering the play through the drift term. But the drift terms are all absorbed by the interest rate.¹⁵ Had the habit level been driven by additional random sources the result will not hold.

Remark 3: literature has shown that the classical Black-Scholes model coincides with a power utility function. Here we shall show that in fact the Black-Scholes model is naturally consistent with market equilibrium. Assume a constant interest rate r and that the state variable S follows a geometric Brownian motion under some probability measure: $dS_t = \mu S_t dt + \sigma S_t dW_t$. It is known that in this case the kernel function G takes the form of

$$G(t, S_t; T, S_T) = e^{-r(T-t)} Z(T) / Z(t) = e^{-r(T-t)} e^{\{-\int_t^T \lambda dW_u - \int_t^T \frac{1}{2} \lambda^2 du\}} = \left(\frac{S_T}{S_t}\right)^{g(t)} e^{-\rho(t)(T-t)}$$

where $Z(t)$ is the Radon-Nikodym derivative

$$\text{process, } g(t) = -\frac{\lambda}{\sigma} \rho(t) = \frac{1}{2} \lambda^2 - \frac{\lambda(\mu - \frac{1}{2} \sigma^2)}{\sigma} + r. \text{ It is straightforward to check that}$$

¹⁵ See equation (4.6) in the Appendix proof.

conditions (2) and (3) are satisfied. The same holds true for the case where the underlying variable follows an arithmetic Brownian motion. These suggest that under risk neutral valuation, some underlying asset price processes are by construction consistent with economic equilibrium while for others market equilibrium imposes additional constraints on the model parameters.

Remark 4: it is well known that in general the risk neutral valuation implies the following pricing partial differential equation (PDE)

$$(0.8) \frac{\partial V}{\partial t} + \frac{1}{2} (\nabla^2 V) \otimes \Sigma + \nabla V \bullet [\bar{\mu} - D\lambda] = rV$$

for multivariate pure diffusion models, where ∇V is the gradient of contingent claim value V with respect to the state variable vector \bar{S} , $\nabla^2 V$ the corresponding Hessian matrix, \otimes a dyadic matrix operator, $\Sigma(t, T; \bar{S})$ the volatility covariance matrix.

Rubinstein (1976), Bollen (1992), and Turvey (2006) derive (3.6) for the univariate risky factor case which includes but does not limit to the Geometric Brownian Motion (GBm), Arithmetic Brownian Motion (ABm), and the Constant Elasticity Volatility (CEV) models. More generally we could have allowed the drift term to depend also on the previous values of the state variable.¹⁶ (3.6) is also derived for Heston stochastic volatility model where the volatility follows a Feller (or Cox-Ingersoll-Ross) process with drift vector $\begin{pmatrix} \mu \\ \alpha(m - Y_t) \end{pmatrix}$, instantaneous volatility covariance

¹⁶ He and Leland (1993) show that as long as the volatility term is only locally dependent on the underlying stock price, then the equilibrium price process must still follow a diffusion process as in (2.1).

$$\text{matrix } \Sigma(t, S_t, Y_t) = \begin{pmatrix} Y_t S_t^2 & \rho \beta Y_t S_t \\ \rho \beta Y_t S_t & \beta^2 Y_t \end{pmatrix}, \text{ Matrix } D = \begin{pmatrix} \sqrt{Y_t} S_t & 0 \\ \rho \beta \sqrt{Y_t} & \beta \sqrt{Y_t} \sqrt{1 - \rho^2} \end{pmatrix}. \text{ Sircar et}$$

al. (2002) for example derived the pricing PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2} V_{ss} Y_t S_t^2 + \frac{1}{2} V_{yy} \beta^2 Y_t + \rho \beta Y_t S_t V_{sy} + r V_s S_t + V_y \left[\alpha(m - Y_t) - \beta \sqrt{Y_t} \left(\rho \frac{\mu - r}{\sqrt{Y_t}} + \sqrt{1 - \rho^2} \lambda_2 \right) \right] = rV$$

On the other hand, conditions (2) and (3) in Proposition 3.1 can be re-written in matrix form: $\tilde{\lambda}_t = -D_t^T \nabla_{T \rightarrow t} G$ and $\nabla_{T \rightarrow t} G \bullet \bar{\mu} + \frac{\partial G}{\partial t} + \frac{1}{2} \nabla_{T \rightarrow t}^2 G \otimes \Sigma = -r$. Now plug in these

two conditions to (3.6) and recall $DD^T = \Sigma$ we have

(0.9)

$$\frac{\partial V}{\partial t} + \frac{1}{2} (\nabla^2 V) \otimes \Sigma + \nabla V \bullet [\bar{\mu} + \Sigma \bullet \nabla_{T \rightarrow t} G] + [\nabla_{T \rightarrow t} G \bullet \bar{\mu} + \frac{\partial G}{\partial t} + \frac{1}{2} \nabla_{T \rightarrow t}^2 G \otimes \Sigma] V = 0$$

Notice that (3.7) is exactly the fundamental pricing differential equation under the multivariate case provided by Garman (1976). There (3.7) is derived through an inter-temporal parity rule. It is important to notice that Garman (1976) makes no assumptions on the market pricing kernel, in particular does not assume risk neutrality. Hence (3.7) is more general than (3.6). Defining the risk free rate on the price space and using (3.7), Garman (1976) derived equation (7) in his dissertation which is essentially condition (3). This suggests that the risk free rate is able to capture many investing characteristics of the representative agent even under more general assumptions such as inter-temporal no arbitrage. Hence a reasonably flexible pricing equation shall be:

$$(0.10) \quad \frac{\partial V}{\partial t} + \frac{1}{2} (\nabla^2 V) \otimes \Sigma + \nabla V \bullet [\bar{\mu} + \Sigma \bullet \nabla_{T \rightarrow t} G] + rV = 0$$

which is more general than (3.6).¹⁷ It will be interesting to compare the pricing performance with real market data across the two pricing equations and design a statistical test of risk neutrality assumption.

Remark 5: all above results are not affected by introducing habit formation so they are very robust with respect to investor preferences. In fact, as long as the market can be completed by some traded assets, these equilibrium conditions will apply. In comparison, the next concept we will discuss, the market risk aversion, is closely related to market investor's preferences.

3.2. Market Risk Aversion

An important advantage of adopting this new definition of MPR is that we can now derive its relationship with MRA. Although the relation between risk aversion and pricing kernel has been studied extensively, there has been no quantitatively specific relation derived for the risk aversion and risk premium.

Since the market is complete, it is well known that the stochastic control problem (3.5) can be transformed to a static optimization problem as follows:

$$(0.11) \quad \begin{aligned} & \max_{\{c_t\}} E \left[\int_0^T u(t, c_t, x_t) dt + U(e_T) \right] \\ & s.t. \quad E \left[\int_0^T G(t) e(t) dt | I_t \right] \leq S_0 \end{aligned}$$

¹⁷ Another implicit model parameter restriction is imposed through martingale restriction (Longstaff (1995)). Setting $\lambda_1 = (\mu - r) / \sqrt{Y_t}$ in condition (2), i.e. the market price of asset risk is equivalent to the Sharpe ratio of the underlying asset, we have an additional constraint.

(3.9) describes a different problem from Jackwerth (2000). The economic environment is not the same and the agent's preference is more general in our case. Detemple and Zapatero (1991) provides its first order condition:

$$(0.10) \quad u_c(\delta(S_t, t), x_t) + k_2 E \left[\int_t^T e^{-k_1(s-t)} u_x(\delta(S_s, s), x_s) ds \middle| \mathbb{F}_t \right] = \varepsilon G(t)$$

where ε is the shadow price for the budget constraint and can be determined from information at time 0. In fact, (3.10) is equivalent to condition (1) in proposition 3.1 where the extra term involving habit level is captured by the value function $J(\cdot)$. On the other hand, a direct calculation of the relative risk aversion from the definition of habit utility shows that

$$(0.11) \quad \gamma_{ts} = - \frac{c(s) u_{cc}(c(s), x(s))}{u_c(c(s), x(s)) + k_2 \left[\int_s^T e^{-k_1(v-t)} u_x(c(v), x(v)) dv \right]}$$

(3.10) and (3.11) together imply an inter-temporal relative risk aversion

$$(0.12) \quad \gamma_{ts} = - \frac{S(s) G_s(t, s)}{G(t, s) + k_2 \varepsilon^{-1} \left\{ \int_s^T e^{-k_1(v-t)} u_x(v) dv - E \left[\int_s^T e^{-k_1(v-t)} u_x(v) dv \middle| \mathbb{F}_s \right] \right\}}$$

Evaluating the above at time t gives the instantaneous relative risk aversion:

$$(0.13) \quad \gamma_t = - \frac{S(t)G_s(t)}{G(t) + k_2 \varepsilon^{-1} \left\{ \int_t^T e^{-k_1(v-t)} [u_x(S_v, x_v) - E[u_x(S_v, x_v) | \mathbb{F}_t]] dv \right\}}$$

A couple of conclusions can be made from (3.13). Under habit formation, the instantaneous market risk aversion is not only dependent on the current pricing kernel and its derivative with respect to the consumption, but also relies on the differential between the future cumulative disutility from the habit level and its expectation. Denote this differential by π . With the regular time-separable von Neumann-Morgenstern preference, π does not appear and the risk aversion $\gamma(t; \bar{S}_t) = -S_t G_s(t; \bar{S}_t) / G(t)$. If the differential is negative, i.e. the future cumulative disutility at time t is less than its expectation, then the agent weighs the disutility more than the current marginal utility of consumption (by (3.10)) hence exhibits higher risk aversion. On the other hand, if the differential is positive, then the agent weighs more on the current marginal utility than on the future disutility hence exhibits less risk aversion, even possibly risk loving. This property permits the market risk aversion under habit formation to vary over time as the differential changes. More importantly, (3.13) poses a potential theoretical challenge to the robustness of the documented pricing kernel puzzle in Jackwerth (2000) and others following the same approach. If the empirical risk aversion is extracted based on the time-separable preference assumption, then the pricing kernel puzzle – locally increasing marginal rate of substitution – may be just a disguised phenomenon of time varying risk aversion.

3.3. Market Prices of Risks and Market Risk Aversion

In equilibrium both market prices of risks and risk aversion are connected with market pricing kernel. Hence we can link the two concepts together by condition (2) in proposition 3.1 and (3.13). Solving G_s from condition (2) in proposition 3.1

gives $G_s / G = -\left((D^T)^{-1} \bar{\lambda}\right)_s'$ where the subscript S indicates the element

corresponding to the underlying asset. Hence the market risk aversion, expressed in terms of the prices of risks, is as the following:

$$(0.14) \quad \gamma_t = -\frac{S_t \left((D^T)^{-1} \bar{\lambda}\right)_s'}{1 + k_2 \varepsilon^{-1} / G(t) \left\{ \int_t^T e^{-k_1(v-t)} [u_x(S_v, x_v) - E[u_x(S_v, x_v) | \mathbb{F}_t]] dv \right\}}$$

Under time-separable preferences, (3.14) becomes

$$(0.15) \quad \gamma_t = S_t \left((D^T)^{-1} \bar{\lambda}\right)_s'$$

which is a *linear* combination of the market price of risks. In the Heston stochastic volatility model for example, it is easily seen that

$$G_s / G = \frac{\rho \lambda_2 / \sqrt{(1 - \rho^2)} - \lambda_1}{\sqrt{Y_t} S_t}$$

Hence the market relative risk aversion is given by

$$(0.16) \quad \gamma_t = \frac{\lambda_1 - \rho \lambda_2 / \sqrt{1 - \rho^2}}{\sqrt{Y_t}}$$

In the benchmark Black-Scholes world there is only one risk factor, the underlying asset. As the kernel function represents the scaled marginal rate of substitution, the marginal utility function is $U'(S) = S^g$. Hence the representative agent's utility function is $U(S) = S^{1+g} / (1+g)$. We know this is a power utility with constant relative risk aversion (CRRA) $-g = \lambda / \sigma$, which coincides with (3.15). This is precisely the risk aversion parameter estimated in Bartunek and Chowdhury (1997). In the arithmetic Brownian motion case the pricing kernel G takes the form of

$$G(t, S_t; T, S_T) = \exp\{-(\lambda / \sigma)(S_T - S_t)\} \exp\{-(r + \lambda \mu / \sigma + 1/2 \lambda^2)(T - t)\}$$

A little algebra then shows that this gives us the exponential utility function with CARA λ / σ , which again is consistent with our derived risk aversion (3.15).

A couple of interesting observations can be made from (3.16).¹⁸ (3.16) implies that the MRA can be written as the Black-Scholes risk aversion plus an extra term in the case of non-zero correlation:

$$(0.17) \quad \gamma_t = \gamma_{BS} + \frac{\lambda_2}{\sqrt{Y_t} / (-\rho / \sqrt{1 - \rho^2})}$$

This extra term is similar to the Black-Scholes term except that the volatility is adjusted by the correlation between the two Brownian motions. The sign of this extra term is also meaningful. By leverage effect as documented in previous studies the correlation coefficient is generally negative. Hence if the market volatility risk premium is positive, which means the market rewards investors for taking the risk due

¹⁸ The conclusions below apply in general to (3.15). We use (3.16) only for illustration purposes.

to volatility randomness, then the implied risk aversion will be higher than the Black-Scholes implied risk aversion level. And vice versa.

Second, *ceteris paribus*, the market risk aversion increases when the market price of asset risk or the market price of volatility risk or both increases. Recall that the market price of asset risk is a function of the first derivative of the pricing kernel which in equilibrium represents the marginal rate of substitution across states. Hence higher market price of asset risk indicates higher absolute value of marginal rate of substitution, which implies higher risk aversion as people demand more hedging securities that provide better return in those “low” states. Similarly, with higher market compensation for taking volatility risk, investors are more risk averse. In addition, the risk aversion is inversely related to the starting volatility level. If the current volatility level is low market players would expect large future volatilities and hence would exhibit bigger risk aversion. On the contrary if the current volatility is already very high, players would not expect too much change in volatility in the future, hence they care less about risk. In the extreme case where volatility is infinite, then everybody in the market does not care about risk at all; everybody appears risk-neutral.

Third, it is worthy to re-emphasize that the market under consideration is complete under any combination of a European option and the underlying stock. Therefore there will be as many completed markets as the number of effectively traded options, or broadly contingent claims. In correspondence to each market there exists a representative agent who exhibits some degree of market risk aversion. Through the market prices of risks we should be able to extract empirical risk aversion from each option with a certain strike and time to maturity. Since the market compensation for

bearing volatility risk has no obvious reason to vary across strikes and time to maturities, the model-implied risk aversion should be constant, implying a flat MRA line. Any deviation from this would either reflect the deficiency of the model not capturing additional risk factors or the transaction costs.

One reason behind linking the market prices of risks with the market risk aversion is that the pricing kernel, as it is associated with indirect utility function, is normally hard to evaluate. The market prices of risks, however, are relatively easier to estimate nowadays from the options market. Hence (3.15) provides a new approach to estimate market risk aversion, which takes into consideration of stochastic volatility.

Section IV – Market Price of Risk and Valuation

Black and Scholes (1973) derived their famous Black-Scholes option pricing formula using both Merton's replication method and Capital Asset Pricing Model (CAPM) argument. Although nowadays the replication portfolio approach is more emphasized in standard option pricing textbooks, the latter CAPM approach actually provides more economic intuition. This section examines option pricing from the angle of CAPM theory and derives some interesting results about various market prices of risks. Assume the following log-normal model:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

Let $c(s,t)$ denote the call price, we have:

$$dc(s, t) = c_s dS_t + c_t dt + \frac{1}{2} c_{ss} dS_t dS_t = c_s dS_t + (c_t + \frac{1}{2} c_{ss} \sigma^2 S_t^2) dt$$

Or equivalently

$$(0.18) \quad \frac{dc_t}{c_t} = \frac{S_t c_s}{c} \frac{dS_t}{S_t} + \frac{c_t + (1/2) c_{ss} \sigma^2 S_t^2}{c} dt$$

From CAPM, the returns of the underlying and the option satisfy:

$$(0.19) \quad E\left[\frac{dS_t}{S_t}\right] - rdt = \beta_{s,m} [E[R_m] - rdt] = \rho_{s,m} \frac{\sigma_{R_s}}{\sigma_{R_m}} [E[R_m] - rdt]$$

$$(0.20) \quad E\left[\frac{dc_t}{c_t}\right] - rdt = \beta_{c,m} [E[R_m] - rdt] = \rho_{c,m} \frac{\sigma_{R_c}}{\sigma_{R_m}} [E[R_m] - rdt]$$

where σ_R denotes the standard deviation of the asset return. These two equations imply that

$$(0.21) \quad \frac{\lambda^c}{\lambda^s} = \frac{\frac{E[dc/c] - rdt}{\sigma_{R_s}}}{\frac{E[dS/S] - rdt}{\sigma_{R_s}}} = \frac{\rho_{c,m}}{\rho_{s,m}} = \frac{\sigma_{R_s}}{\sigma_{R_c}} \frac{S c_s}{c}$$

We use λ^c to denote the Sharpe ratio of the option and λ^s being the Sharpe ratio of the underlying asset. But from (5.1) we see that the return of the underlying and the return of the option are perfectly correlated, hence the correlation coefficient ratio on the

right hand side of (5.4) is exactly one. Denote $\xi^s = \frac{S C_s}{C}$ as the elasticity of the option price with respect to the underlying. The perfect correlation between the return of option and the return of the underlying implies that the elasticity simply measure the relative volatility of option with respect to the underlying, which is always greater than one. Put it another way, the ratio of the elasticity and the relative volatility is exactly one. It turns out that this ratio is very important in quantifying the relationship among the market price of risks we are interested in and hence deserve a notation. We use τ to denote this ratio throughout this section.

The above reasoning gives:

$$(0.22) \quad \lambda^c = \frac{E[dc / c] - rdt}{\sigma_{R_c}} = \frac{E[dS / S] - rdt}{\sigma_{R_s}} = \lambda^s$$

Now from (4.5) with a little bit of algebra we get the standard Black-Scholes Partial Differential equation. (4.5) is simply saying that in capital market equilibrium the Sharpe ratio of any two risky assets in the market that share a common source of risk has to be the same. It is crucial to note that in order for this statement to be true, the assets have to be liquidly traded in the market. Figure 2.1 shows graphically the meaning of equation (4.5):

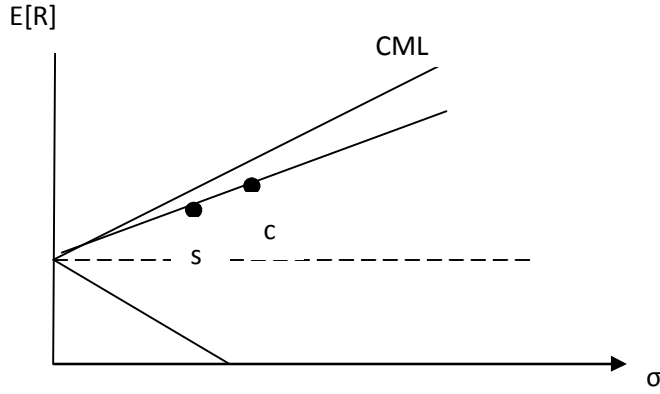


Figure 1: Sharpe Ratios for Two Risky Assets with Same Random Source of Risk

The slope of the Capital Market Line (CML) is the Sharpe ratio of the market portfolio—traditionally termed as the “market price of risk”. By (4.2) or (4.3) the Sharpe ratio of every risky asset has to be at most equal to the market price of risk of the market portfolio. If two assets are driven by the same randomness, then their Sharpe ratios have to be the same, hence lying on the same line.

For practical purposes we are usually more interested in a multivariate model. One asset’s price sometimes is determined by more than one *traded* state variable. To illustrate the relations among assets’ Sharpe ratios under this setting, we first consider a two-asset option pricing problem. Assume the underlying assets follow the general geometric Brownian motions as follows:

$$\begin{aligned}
 dS_{1t} &= \mu_{1t}S_{1t}dt + \sigma_{1t}S_{1t}dW_t \\
 dS_{2t} &= \mu_{2t}S_{2t}dt + \sigma_{2t}S_{2t}(\rho_t dW_t + \sqrt{1-\rho_t^2} dZ_t) \\
 dZ_t dW_t &= 0
 \end{aligned}
 \tag{0.23}$$

where S_1 is the first underlying asset, S_2 the second underlying asset. Let the price of the call option written on this asset be $c(t, s_1, s_2)$. Using Ito's formula, we have:

$$\begin{aligned}
 (0.24) \quad dc(t, s_1, s_2) &= c_t dt + c_{s_1} dS_{1t} + c_{s_2} dS_{2t} + \frac{1}{2} c_{s_1 s_1} dS_{1t} dS_{1t} + \frac{1}{2} c_{s_2 s_2} dS_{2t} dS_{2t} + c_{s_1 s_2} dS_{1t} dS_{2t} \\
 &= c_{s_1} dS_{1t} + c_{s_2} dS_{2t} + (c_t + \frac{1}{2} c_{s_1 s_1} \sigma_{1t}^2 s_{1t}^2 + \frac{1}{2} c_{s_2 s_2} \sigma_{2t}^2 s_{2t}^2 + c_{sy} \rho_t \sigma_{1t} \sigma_{2t} s_{1t} s_{2t}) dt
 \end{aligned}$$

We can rewrite it as:

$$(0.25) \quad \frac{dc(t, s_1, s_2)}{c} = \frac{S_{1t} c_{s_1}}{c} \frac{dS_{1t}}{S_{1t}} + \frac{s_{2t} c_{s_2}}{c} \frac{dS_{2t}}{S_{2t}} + \frac{(c_t + \frac{1}{2} c_{s_1 s_1} \sigma_{1t}^2 s_{1t}^2 + \frac{1}{2} c_{s_2 s_2} \sigma_{2t}^2 s_{2t}^2 + c_{sy} \rho_t \sigma_{1t} \sigma_{2t} s_{1t} s_{2t})}{c} dt$$

Since both the two underlyings and the option written on them are assets *traded* in the market, CAPM applies to all of them:

$$(0.26) \quad \frac{\lambda^c}{\lambda^1} = \frac{\rho_{c,m}}{\rho_{s_1,m}}$$

$$(0.27) \quad \frac{\lambda^c}{\lambda^2} = \frac{\rho_{c,m}}{\rho_{s_2,m}}$$

where $\rho_{s,m}$ is the correlation between asset S and the market portfolio. From (4.8), (4.9) and (4.10) we get the following relation among the three market prices of risks:

$$(0.28) \quad \lambda^c = \left(\frac{\sigma_{R_1}}{\sigma_{R_c}} \xi^1 \right) \lambda^1 + \left(\frac{\sigma_{R_2}}{\sigma_{R_c}} \xi^2 \right) \lambda^2$$

Or in terms of the ratios we defined previously,

$$(0.29) \quad \lambda^c = \tau^1 \lambda^1 + \tau^2 \lambda^2$$

Equation (4.12) says that the market price of risk for the option can be thought of as a weighted sum of the two market prices of risks of the two underlying assets. The weights are the ratios of the elasticity of option with respect to the underlying and the relative volatility. But unlike the previous single-variable case, the variation of the return on option is now split into three parts: variation due to the first underlying, variation due to the second underlying and a correction from the correlation between the two underlying processes. Therefore, we should not expect the relative volatility of option with respect to each underlying to equal the corresponding elasticity. In fact, from (4.6) and (4.8) we have:

$$\begin{aligned} \sigma_{R_1} &= \sigma_1 \sqrt{dt} \\ \sigma_{R_2} &= \sigma_2 \sqrt{dt} \\ \sigma_{R_c} &= \sqrt{(\xi^1 \sigma_1)^2 + (\xi^2 \sigma_2)^2 + 2\rho \xi^1 \xi^2 \sigma_1 \sigma_2} \sqrt{dt} \end{aligned}$$

We look at three special cases corresponding to the correlation coefficient being 1, 0, and -1. Suppose the two underlying are perfectly correlated, then $\tau^1 + \tau^2 = 1$. Since both are strictly positive, the two ratios are strictly less than one. In this case the Sharpe ratio of the option is just a linear combination of the two asset Sharpe ratios. If

the two underlying processes are independent of each other, then $(\tau^1)^2 + (\tau^2)^2 = 1$. Again the two ratios are less than one. If the two underlying are perfectly negatively correlated, then $\tau^2 = \tau^1 - 1$ if $\tau^1 > \tau^2$. Therefore the first ratio is always greater than one. Similarly, $\tau^2 = \tau^1 + 1$ if $\tau^1 < \tau^2$. In this case the second ratio is always greater than one. Figure 2.2 depicts the relation between the two ratios in these three cases:

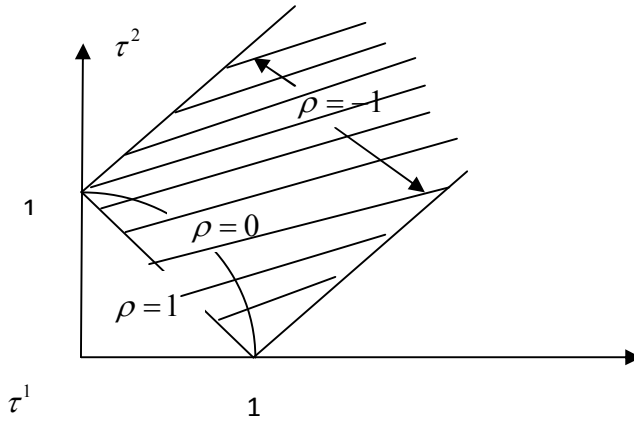


Figure 2 - Relation between Elasticities for Underlying Asset and Option

Section V – Conclusion

The theoretical results presented in this chapter can be used for several empirical works. One can test the risk-neutrality assumption by comparing the restricted model of (3.6) and the unrestricted model of (3.8). One can also empirically extract market risk aversion through market prices of risks in stochastic volatility models. It is our expectation that with models more adequately capturing the real dynamics of the underlying stochastic processes and more risk factors being incorporated into the model the implied market prices of risks and risk aversion shall behave closer to flat

lines across strikes and maturities. On the theoretical part, one can examine the equilibrium relations between MPR, MRA and pricing kernel under the case of jump-diffusion models (see Bates (1996), Chernov et.al (2003), Carr and Sun (2007)).

APPENDIX 2.1

PROOF OF PROPOSITION 3.1

Recall that the system of risk factor and asset price processes are given as the following:

$$\begin{aligned}
dS_t &= (\mu_1 - D)dt + \sigma_1 dW_{1t} \\
d\varsigma_{2t} &= \mu_2 dt + \rho \sigma_2 dW_{1t} + \sqrt{1 - \rho^2} \sigma_2 dW_{2t} \\
dV_t &= \mu_3 dt + F d\hat{W}_t \\
dW_{1t} dW_{2t} &= 0, d\hat{W}_t dW_{1t} = \rho' dt \\
de_t &= adt + bdW_{1t} + cdW_{2t}, \\
\text{where } a &= \alpha \frac{\mu_1}{S} + \beta \frac{\mu_3}{V} + (e - \alpha - \beta)r, b = \alpha \frac{\sigma_1}{S} + \rho' \beta \frac{F}{V}, c = \sqrt{1 - \rho'^2} \beta \frac{F}{V}
\end{aligned}$$

and S_t is the asset price, D the dividend, ς_t the second risk factor, V_t the put option price, e_t the wealth at time t . Let $J(t, e_t, S_t, V_t, x_t)$ be the value function for problem (2.7). Then Hamilton-Jacobi-Bellman equation gives:

$$0 = \max_{(\alpha, \beta)} \left[u(t, c_t, x_t) + E_t \left(\frac{dJ}{dt} \right) \right]$$

Apply Ito's lemma to $J(t, e_t, S_t, V_t, x_t)$ and plug in the stochastic differential equations for the underlying asset, the put option and the wealth, we have:

(0.30)

$$0 = u(c, x) + J_t + aJ_e + (\mu_1 - D)J_S + \mu_3 J_V + J_x(k_2 c - k_1 x) + \frac{1}{2} J_{ee}(b^2 + c^2) + \frac{1}{2} J_{ss} \sigma_1^2 + \frac{1}{2} J_{VV} F^2 + b\sigma_1 J_{Se} \\ + (\beta \frac{F^2}{V} + \alpha \frac{\sigma_1 F}{S} \rho') J_{Ve} + \sigma_1 F \rho' J_{SV}$$

First order conditions with respect to c , α and β are:

(0.31)

$$0 = u_c(c, x, t) + k_2 J_x - J_e \quad (A.1)$$

$$0 = \left(\frac{\mu_1 - D}{S} - r \right) J_e + \frac{\sigma_1}{S} \left(\frac{\alpha \sigma_1}{S} + \rho' \frac{\beta F}{V} \right) J_{ee} + \frac{\sigma_1^2}{S} J_{Se} + \frac{\sigma_1 F \rho'}{S} J_{Ve} \quad (A.2)$$

$$0 = \left(\frac{\mu_3}{V} - r \right) J_e + \left[\left(\frac{\alpha \sigma_1}{S} + \rho' \frac{\beta F}{V} \right) \frac{\rho' F}{V} + (1 - \rho'^2) \frac{\beta F^2}{V^2} \right] J_{ee} + \rho' \frac{\sigma_1 F}{V} J_{Se} + \frac{F^2}{V} J_{Ve} \quad (A.3)$$

Now, differentiating (3.18) with respect to e we have:

(0.32)

$$0 = J_{te} + rJ_e + aJ_{ee} + (\mu_1 - D)J_{Se} + \mu_3 J_{Ve} + J_{xe}(k_2 c - k_1 x) + \frac{1}{2} J_{eee}(b^2 + c^2) + \frac{1}{2} J_{sse} \sigma_1^2 + \frac{1}{2} J_{VVe} F^2 + b\sigma_1 J_{See} \\ + (\beta \frac{F^2}{V} + \alpha \frac{\sigma_1 F}{S} \rho') J_{Vee} + \sigma_1 F \rho' J_{SVe}$$

This suggests that dJ_e has a drift term $-rJ_e dt$ and its diffusion terms are¹⁹

$bJ_{ee} + \sigma_1 J_{Se} + \rho' F J_{Ve}$ for dW_{1t} and $cJ_{ee} + \sqrt{1 - \rho'^2} F J_{Ve}$ for dW_{2t} . That is,

¹⁹ Apply Ito's Lemma to the function $e^{rt} J_e(t, e_t, S_t, V_t, x_t)$ and use (3.20) for the drift term.

$$dJ_e = -rJ_e dt + (bJ_{ee} + \sigma_1 J_{Se} + \rho' FJ_{Ve})dW_{1t} + (cJ_{ee} + \sqrt{1-\rho'^2} FJ_{Ve})dW_{2t}$$

We claim that the above can be rewritten as:

$$(0.33) dJ_e = -rJ_e dt - \lambda_1 J_e dW_{1t} - \lambda_2 J_e dW_{2t}$$

To see this, recall that the discounted underlying asset and discounted put option value are both martingales under the risk-neutral measure defined through market prices of risks. For the former, we have

$$(0.34) \lambda_{1t} = \frac{\mu_{1t} - D_t - rS_t}{\sigma_{1t}}$$

For the latter we have

$$(0.35) \frac{\mu_{3t} - rV_t}{F_t} = \rho' \lambda_1 + \sqrt{1-\rho'^2} \lambda_2$$

Note that the left hand is the market price of option risk, while the right hand side is a linear combination of the market price of asset risk and the market price of volatility risk. Hence (3.23) provides a nice relationship between the three MPRs.

Now we plug (3.22) into equation (A.2) in (3.19) to get:

$$(0.36) -\lambda_1 J_e = bJ_{ee} + \sigma_1 J_{Se} + \rho' FJ_{Ve}$$

We write equation (A.3) in (3.19) by adding and subtracting the same term as:

$$(\mu_3 - rV)J_e + bF\rho'J_{ee} + (1 - \rho'^2)\frac{\beta F^2}{V}J_{ee} + \sigma_1 F\rho'J_{Se} + \rho'^2 F^2 J_{Ve} - \rho'^2 F^2 J_{Ve} + F^2 J_{Ve} = 0$$

Using (3.23) the above is equivalent to

$$(\mu_3 - rV - \lambda_1 F\rho')J_e + (1 - \rho'^2)\frac{\beta F^2}{V}J_{ee} + F^2(1 - \rho'^2)J_{Ve} = 0$$

Notice that the bracket term in the J_e term equals $F\sqrt{1 - \rho'^2}\lambda_2$ by (3.23), plug in and rearrange we have

$$(0.37) \quad -\lambda_2 J_e = cJ_{ee} + F\sqrt{1 - \rho'^2}J_{Ve}$$

(3.24) and (3.25) together imply (3.21), which is equivalent to

$$J_e(t, e_t, S_t, V_t) = J_e(0) \exp \left\{ -rt - \int_0^t \lambda_{1s} dW_{1s} - \frac{1}{2} \int_0^t \lambda_{1s}^2 ds - \int_0^t \lambda_{2s} dW_{2s} - \frac{1}{2} \int_0^t \lambda_{2s}^2 ds \right\}$$

which implies $\frac{J_e(t, e_t, S_t, V_t)}{J_e(0)} = G(t, S_t, V_t)$

Hence there exists a (indirect) utility function that supports the pricing kernel. Plug in (A.1) in (3.19) and use the equilibrium condition that consumption equals the dividends we have (1) in the proposition. The terminal condition comes from the fact

that at the terminal period the agent will not care about the dividends and the volatility and just consume the underlying asset.

To derive condition (2) in the proposition, we define a function

$$H(t, S_t, \varsigma_t, x_t) = E \left[e^{r(T-t)} U'(S_T) | S_t, \varsigma_t, x_t \right] = J_e(0) E \left[e^{r(T-t)} G(S_T) | S_t, \varsigma_t, x_t \right] = J_e(0) G(t, S_t, \varsigma_t, x_t)$$

The last equivalence comes from the fact that $e^{rt} G(t)$ is a martingale. Now applying Ito's lemma to $H(t, S_t, \varsigma_t, x_t)$ and comparing its volatility terms to the market prices of risks embodied in the definition of the pricing kernel give the desired result. By comparing the drift terms in $H(t, S_t, \varsigma_t, x_t)$ and $G(t, S_t, \varsigma_t, x_t)$ we can derive condition (3).

□*Q.E.D.*

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CHAPTER 3

A FINITE DIFFERENCE METHOD FOR SOLVING STOCHASTIC VOLATILITY MODELS

As aforementioned, one purpose of this dissertation is to back out the implied market prices of risk from option prices. In order to do this, we first need to be able to solve pricing differential equation or more generally any Convection-Diffusion partial differential equations. As there is usually no closed form solutions we have to apply numerical techniques such as Monte Carlo simulation (MC) or Finite Difference methods (FDM) or Finite Element methods (FEM). We choose finite difference method here as it is widely used in research as well as in industry. The popular trinomial tree option pricing method is equivalent to explicit scheme, a special implementation of FDM. We break this section into four parts. First, we introduce the basic idea and some necessary notations involved in the FDM; second, we apply FDM to a one-factor parabolic PDE and use classic Black-Scholes equation as an application; third, we demonstrate how to use splitting operator technique combined with Yanenko's scheme to solve a two-factor PDE as described in Duffy (2005), then we use Heston's stochastic volatility model as an example. We should state at this point that Duffy (2005) merely suggested that Yanenko's scheme could be used to solve Heston SVM, but we are unaware of any empirical study other than ours that has actually accomplish this; fourth, we adopt the penalty method as discussed in Duffy (2005) to solve for a free boundary PDE problem and use the American option as an example. Again, while Duffy (2005) provides the theoretical condition for the free boundary problem, we implement his approach and provide empirical validation of his propositions. We are unaware of any empirical studies to have applied his approach to

the pricing of American options; fifth, the relation between various model parameters and implied volatility smiles are demonstrated by numerical experiment using the FDM discussed in this section. Along with our discussion the boundary conditions for each problem are specified. The computer code in VBA is attached in the appendices. The last section concludes the chapter.

Section I - Idea of FDM and Notations

The idea of FDM is to approximate the derivative terms in a PDE by time and space discretizations. Consider a function $f(x)$ on some domain D . We can approximate its first order derivative $f'(x_n)$ at point x_n by the following:

$$\begin{aligned} (1) \quad D_+ f^n &= \frac{f^{n+1} - f^n}{k} \text{-----forward difference} \\ (2) \quad D_- f^n &= \frac{f^n - f^{n-1}}{k} \text{-----backward difference} \\ (3) \quad D_0 f^n &= \frac{f^{n+1} - f^{n-1}}{2k} \text{-----central difference} \end{aligned}$$

The second order derivative is approximated by:

$$(4) \quad D_+ D_- f^n = \frac{f^{n+1} - 2f^n + f^{n-1}}{k^2}$$

where f^n denotes the function value at x_n , k is the mesh size in the x direction. By Taylor series analysis we see that the central difference formula (3) and the second order derivative approximation (4) are second order approximations to their respective

derivative if k is small enough, while the two one-sided formulae (1) and (2) are just first order approximations.

Now we consider a simple one factor hyperbolic PDE and show how to discretize it:

$$f'(t) + a(t)f(t) = g(t)$$

Three common schemes for this PDE are available:

$$(5) \quad D_+ f^n + a^{n+1} f^{n+1} = g^{n+1} \text{-----Implicit Euler Scheme}$$

$$(6) \quad D_+ f^n + a^n f^n = g^n \text{-----Explicit Euler}$$

Scheme

$$(7) \quad D_+ f^n + a^{n+1/2} \frac{f^n + f^{n+1}}{2} = g^{n+1/2} \text{-----Crank-Nicolson}$$

Scheme

$$\text{where } a^{n+1/2} = a(t_{n+1/2}), \quad g^{n+1/2} = g(t_{n+1/2}), \quad t_{n+1/2} = \frac{1}{2}(t_n + t_{n+1})$$

It turns out that the Crank-Nicolson scheme is second order accurate while the implicit and explicit schemes are only first order accurate. That is part of the reason why Crank-Nicolson has been so popular in the financial engineering literature. However, as Duffy (2005) pointed out, this scheme can sometimes produce spurious oscillations near the strike price. On the other hand, accuracy can always be improved for implicit and explicit schemes by use of extrapolation. From (5)-(7) we also see that for the explicit scheme we can always solve for function values at time $n+1$ directly from previous values at time n , but have to rely on some matrix equation solver for implicit

and Crank-Nicolson schemes. As shown in Duffy (2005), usually we need some regularity conditions to ensure stability and convergence for explicit schemes, but we always have unconditional stability and convergence for the implicit scheme. For this reason, this Dissertation will use implicit schemes whenever possible although at the cost of longer computational time.

Section II - FDM for One-Factor Parabolic PDE and its Application to Black-Scholes PDE

In this section we consider a one-factor parabolic PDE in the following general form:

(8)

$$\begin{aligned} -\frac{du}{dt} + \sigma(x,t)\frac{d^2u}{dx^2} + \mu(x,t)\frac{du}{dx} + b(x,t)u &= f(x) \\ u(x,0) &= \varphi(x), x \in \Omega \\ u(A,t) &= g_0(t), u(B,t) = g_1(t), t \in (0,T) \end{aligned}$$

We call $\sigma(x,t)$ the diffusion term of the PDE, $\mu(x,t)$ the convection term, $b(x,t)$ the zero term, and $f(x)$ the forcing term. Initial condition at $t=0$ and boundary conditions at $x=A$ and $x=B$ are also given as above.

If we break the state variable domain into J intervals with mesh points $x_0, x_1, x_2, \dots, x_J$ and the time domain into N intervals, then the corresponding FDM scheme is:

(9)

$$-\frac{u_j^{n+1} - u_j^n}{k} + \sigma_j^{n+1} D_+ D_- u_j^{n+1} + \mu_j^{n+1} D_0 u_j^{n+1} + b_j^{n+1} u_j^{n+1} = f_j^{n+1}$$

$$u_j^0 = \varphi(x_j) \text{ for } j = 1, \dots, J-1$$

$$u_0^n = g_0(t_n) \text{ for } n = 0, 1, \dots, N$$

$$u_J^n = g_1(t_n)$$

In practice an exponential fitting factor is usually used to replace the original diffusion term to handle discontinuities near strike prices. The factor is defined as

$$(10) \quad \gamma = \frac{\mu h}{2} \coth \frac{\mu h}{2\sigma}$$

where h is mesh size in the state variable dimension and $\coth(x) = \frac{e^{2x} + 1}{e^{2x} - 1}$. By lemma 1 and lemma 2 in chapter 18 in Duffy (2005), this fully implicit scheme is uniformly stable, converges, and is oscillation-free.

Using the above results which are due to Duffy (2005) we further implement the scheme in the following way to form the mathematical or computational basis for computer implementation. The scheme can be written explicitly as:

$$(11) \quad -\frac{u_j^{n+1} - u_j^n}{k} + \gamma_j^{n+1} \left[\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h^2} \right] + \mu_j^{n+1} \frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2h} + b_j^{n+1} u_j^{n+1} = f_j^{n+1}$$

Collecting terms we get:

(12)

$$A_j^{n+1}u_{j-1}^{n+1} + B_j^{n+1}u_j^{n+1} + C_j^{n+1}u_{j+1}^{n+1} = F_j$$

$$\text{where } A_j^{n+1} = \frac{k\gamma_j^{n+1}}{h^2} - \frac{k\mu_j^{n+1}}{2h}$$

$$B_j^{n+1} = -1 - \frac{2k\gamma_j^{n+1}}{h^2} + kb_j^{n+1}$$

$$C_j^{n+1} = \frac{k\gamma_j^{n+1}}{h^2} + \frac{k\mu_j^{n+1}}{2h}$$

$$F_j = kf_j^{n+1} - u_j^n$$

To develop a computer algorithm we define a vector $\bar{u} = (u_1^{n+1}, \dots, u_{J-1}^{n+1})$, a matrix

$$\bar{A}^{n+1} = \begin{pmatrix} B_1^{n+1} & C_1^{n+1} & \mathbf{0} \\ A_2^{n+1} & \ddots & \vdots \cdot C_{J-1}^{n+1} \\ \mathbf{0} & \vdots \cdot A_{J-1}^{n+1} & B_{J-1}^{n+1} \end{pmatrix} \text{ and a vector } \bar{G} = \begin{pmatrix} kf_1^{n+1} - u_1^n - a_1^{n+1}u_0^{n+1} \\ kf_2^{n+1} - u_2^n \\ \vdots \\ kf_{J-2}^{n+1} - u_{J-2}^n \\ kf_{J-1}^{n+1} - u_{J-1}^n - c_{J-1}^{n+1}u_J^{n+1} \end{pmatrix} \text{ so that our}$$

derivatives pricing scheme can be rewritten in a matrix form:

$$\bar{A} \bullet \bar{u} = \bar{G}$$

Notice that matrix A is a tri-diagonal matrix. Hence with initial condition and boundary conditions we can recursively solve for the vector \bar{u} at each time step using standard algorithm such as the LU Decomposition.

The application to the Black-Scholes PDE is straight forward. All we need to change are the coefficients in the PDE (8) to reflect Brownian volatility and risk-neutral drift. Although we do not present Black-Scholes result in this chapter, we have confirmed

that our approach to pricing such derivatives using (12) will converge to the Black-Scholes price within (to our precision settings) 1/1000 of a dollar.

Section III - FDM for Two-Factor Parabolic PDE and its Application to Heston Model

A more important and less trivial application of the approach is to the Heston SVM. The major reference of this section is Daniel Duffy (2006) and Yanenko (1971). We will mainly focus on the Heston PDE, but the approach is applicable to more general PDEs. The Heston model is important because it includes even within a tradable security and derivative security a non-market priced factor. That is even when applied to conventional products the inability of the market to use existing securities to span volatility risk requires recognition of the market price of risk. In addition, the Heston SVM is more popular both in practice and in academic research because it fits the market data well by capturing the excess kurtosis of the underlying asset distribution through stochastic volatility. Consider a Heston PDE:

$$(13) \quad -\frac{\partial U}{\partial t} + \frac{1}{2}U_{ss}Y_tS_t^2 + \frac{1}{2}U_{yy}\beta^2Y_t + \rho\beta Y_tS_tU_{sy} + rU_sS_t + U_y[\alpha(m - Y_t) - \lambda(s, y, t)] = rU$$

with boundary conditions corresponding to the case where $S=0$, $S=\infty$, $y=0$, and $y=\infty$ respectively:

$$(13.a) U(0, y, t) = 0$$

$$(13.b) U_s(\infty, y, t) = 1$$

$$(13.c) U(s, \infty, t) = s$$

$$(13.d) -U_t + rsU_s + \alpha mU_y = rU$$

Notice that when the underlying asset has price 0, the derivative based on it has value 0 also. When the underlying asset price goes to infinity, the derivative also has value going to infinity such that in the limit the delta approaches one. On the other hand, when volatility goes to infinity the derivative has the same price as the underlying, while when volatility goes to zero we have a PDE as in (13.d).

Using (1) to (4) we approximate the derivatives as follows:

$$\begin{aligned}\Delta_x^2 U_{i,j} &= \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} \\ \Delta_y^2 U_{i,j} &= \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{h^2} \\ \Delta_x U_{i,j} &= \frac{U_{i+1,j} - U_{i-1,j}}{2h} \\ \Delta_y U_{i,j} &= \frac{U_{i,j+1} - U_{i,j-1}}{2h} \\ \Delta_x \Delta_y U_{i,j} &= \frac{U_{i+1,j+1} - U_{i-1,j+1} - U_{i+1,j-1} + U_{i-1,j-1}}{4h^2}\end{aligned}$$

Now, under Yanenko (1971), the splitting scheme will be as follows:

$$-\frac{U_{i,j}^{n+1/2} - U_{i,j}^n}{k} + A_{i,j}^{n+1/2} \Delta_x^2 U_{i,j}^{n+1/2} + B_{i,j}^{n+1/2} \Delta_x U_{i,j}^{n+1/2} + C_{i,j}^{n+1/2} U_{i,j}^{n+1/2} + \frac{1}{2} F_{i,j}^n \Delta_x \Delta_y U_{i,j}^n = 0 \quad (14.a)$$

$$-\frac{U_{i,j}^{n+1} - U_{i,j}^{n+1/2}}{k} + D_{i,j}^{n+1} \Delta_y^2 U_{i,j}^{n+1} + E_{i,j}^{n+1} \Delta_y U_{i,j}^{n+1} + \frac{1}{2} F_{i,j}^{n+1/2} \Delta_x \Delta_y U_{i,j}^{n+1/2} = 0 \quad (14.b)$$

where

$$A = \frac{1}{2} Y S^2, B = rS, C = -r, D = \frac{1}{2} Y \beta^2, E = \alpha(m - Y) - \lambda, F = \rho \beta Y S$$

the subscripts are dropped for brevity.

For the first split scheme (14.a), after plugging in the approximations and rearranging the terms, we have (15.a):

$$\overline{A}_{i,j}^{n+1/2} U_{i-1,j}^{n+1/2} + \overline{B}_{i,j}^{n+1/2} U_{i,j}^{n+1/2} + \overline{C}_{i,j}^{n+1/2} U_{i+1,j}^{n+1/2} = G^n$$

where

$$\begin{aligned}\overline{A}_{i,j}^{n+1/2} &= \frac{k}{h^2} \gamma_{i,j}^{n+1/2} - \frac{B_{i,j}^{n+1/2} k}{2h} \\ \overline{B}_{i,j}^{n+1/2} &= -1 - \frac{2k}{h^2} \gamma_{i,j}^{n+1/2} + k C_{i,j}^{n+1/2} \\ \overline{C}_{i,j}^{n+1/2} &= \frac{k}{h^2} \gamma_{i,j}^{n+1/2} + \frac{B_{i,j}^{n+1/2} k}{2h}\end{aligned}$$

for $i=1,2,\dots,I-1, j=1,2,\dots,J-1$

Similarly, the split scheme for (14.b) is (15.b):

$$\overline{A}_{i,j}^{n+1} U_{i,j-1}^{n+1} + \overline{B}_{i,j}^{n+1} U_{i,j}^{n+1} + \overline{C}_{i,j}^{n+1} U_{i,j+1}^{n+1} = G^{n+1/2}$$

where

$$\begin{aligned}\overline{A}_{i,j}^{n+1} &= \frac{k}{h^2} D_{i,j}^{n+1} - \frac{E_{i,j}^{n+1} k}{2h} \\ \overline{B}_{i,j}^{n+1} &= -1 - \frac{2k}{h^2} D_{i,j}^{n+1} \\ \overline{C}_{i,j}^{n+1} &= \frac{k}{h^2} D_{i,j}^{n+1} + \frac{E_{i,j}^{n+1} k}{2h}\end{aligned}$$

for $i=1,2,\dots,I-1, j=1,2,\dots,J-1$

As before the D factor will be replaced by its corresponding exponential fitting factor as defined in (10). The boundary conditions are straightforward as defined in (13.a) to (13.c). (13.d) is a little more complex because as we see in that case the PDE (13) reduces to a two-factor one-order hyperbolic PDE:

$$-U_t + rsU_s + \alpha mU_y = rU$$

We can again apply operator splitting method to this PDE to numerically solve it. The steps are similar to (13), (14.a) and (14.b), (15.a) and (15.b). Now we have the full schemes and initial and boundary conditions ready. So we can solve the PDE recursively. We developed a VBA program implementing the above algorithm. The code is attached in the Appendix of this chapter. Except for the LU decomposition algorithm, all other codes are written by me. The results are satisfying compared with those in previous literature. Table 1 compares our result with those obtained by others for the same sets of parameters. One can see that the results are very close to each other.

Table 1 - European Call Prices Calculated with Finite Difference Methods

interest rate	r	0.0000000	0.0536877	0.1000000
dividend rate	d	0.0000000	0.0000000	0.0000000
Strike	K	100.0000000	1050.0000000	123.4000000
Maturity	T	0.500000	1.0000000	1.0000000
Volatility of volatility	σ	0.1000000	0.1136000	0.1500000
mean reversion rate	κ	2.0000000	3.2489000	1.9889370
mean reversion level	θ	0.0100000	0.7742240	0.0118760
correlation coefficient	ρ	0.0000000	-0.3372000	-0.9000000
spot price	S	100.0000000	1268.2100000	123.4000000
Volatility	v	0.0100000	0.0299982	0.0200000
drift rate	μ	0.0000000	0.0000000	0.0000000
market price risk	λ	0.0000000	0.0000000	0.0000000
Analytic Solution		2.7911624	481.2795631	13.8571000
My Solution		2.7938022	481.5630	13.83640

FDM parameters used: minimum volatility=0, maximum volatility=0.5, starting time t=0, ending time T=0.5, number of steps in the underlying stock direction=327, number of steps in the volatility direction=327, number of steps in the time dimension =600. Notice that here we do not differentiate between market price of asset risk and market price of volatility risk since we set both to zero following the convention in the literature. Analytic solution is obtained by the closed form solution to Heston SVM as in Heston (1993).

Section IV - FDM for Free Boundary PDE and its Application to American Option Pricing

Pricing American option numerically is slightly more complex. Various methods have been proposed to solve this problem (Nielsen et. (2002), Aitsahlia, Goswami & Guha (2008), Chockalingam & Muthuraman (2007), Zvan, Forsyth & Vetzal (1997), Ikonen & Toivanen (2004), Ikonen & Toivanen (2007)). Front fixing by Landon transformation, for example, transforms the original PDE with free boundary into a new PDE with fixed domain, thus allowing us to apply the regular FDM. But front fixing is only good for one dimensional problems, hence not a choice for this dissertation. Another idea is to add a penalty term to the original PDE, this way we really do not need to bother about the free boundary at all. On the other hand, simplicity is traded for heavier time-consumption.

In general, the penalty method removes the free boundary by adding a small, continuous penalty term to the PDE as:

(16)

$$\frac{\partial P(\varepsilon)}{\partial t} + LP(\varepsilon) + f(P(\varepsilon)) = 0$$

$$\text{usually } f(P(\varepsilon)) = \frac{\varepsilon C}{P(\varepsilon) + \varepsilon - q(S)}, q(S) = K - S \text{ and } C \geq rK.$$

K is the strike price, S the asset price, P the put price.

Here we adopt the semi-implicit method which in a 1-factor PDE is as follows:

$$\frac{P_j^{n+1} - P_j^n}{k} + LP_j^n + f_j^{n+1}(P_j^{n+1}) = 0$$

$$\text{where } LP_j^n = \frac{1}{2} \sigma^2 S_j^2 D_+ D_- P_j^n + r S_j D_0 P_j^n - r P_j^n$$

Of course we have additional constraint:

$$P_j^n \geq (K - S_j)^+ \text{ for all } j$$

It is shown in Duffy that this constraint is satisfied if the time step size

$$k \leq \frac{\varepsilon}{rK}$$

Now, consider the Heston PDE with penalty term $f_\varepsilon(U)$:

$$\frac{\partial U}{\partial t} + \frac{1}{2} U_{ss} Y_t S_t^2 + \frac{1}{2} U_{yy} \beta^2 Y_t + \rho \beta Y_t S_t U_{sy} + r U_s S_t + U_y [\alpha(m - Y_t) - \lambda(s, y, t)] + f_\varepsilon(U) = rU$$

By operator splitting we have

$$\frac{\partial U}{\partial t} + \frac{1}{2} Y S^2 \frac{\partial^2 U}{\partial S^2} + r S \frac{\partial U}{\partial S} - rU + f_\varepsilon(U) + \rho \beta Y S \frac{\partial^2 U}{\partial Y \partial S} = 0 \quad (17.a)$$

$$\frac{\partial U}{\partial t} + \frac{1}{2} \beta^2 Y \frac{\partial^2 U}{\partial Y^2} + (\alpha(m - Y) - \lambda Y) \frac{\partial U}{\partial Y} = 0 \quad (17.b)$$

Discretize the above we get almost identical difference equations as in part III except that in the first step equation we add an addition term due to the penalty,

$-\frac{\varepsilon k C}{U_{i,j}^{n+1} + \varepsilon - q}$. The second step equation is unchanged. Therefore, little change in the

algorithm is needed in pricing American options.

Results are compared against benchmarks in the literature such as Chocklingam and Muthuraman, where the parameters are as follows:

$$K = 10, r = 0.1, q = 0, \sigma = 0.9, \alpha = 5, \rho = 0.1, m = 0.16, T = 0.25.$$

In Table 2 we compare our results with others and they are again very close to each other.

Table 2 - Finite Difference American Put Prices Compared with Other Methods

		Spot Prices				
Initial Vol		8	9	10	11	12
0.0625	IT	2.000	1.108	0.520	0.214	0.082
	FDM	2.000	1.108	0.519	0.214	0.082
0.25	Sim	2.127	1.347	0.796	0.446	0.235
	CM	2.074	1.325	0.785	0.440	0.238
	FDM	2.075	1.330	0.792	0.444	0.240
0.375	Sim	2.175	1.448	0.928	0.573	0.343
	CM	2.149	1.448	0.928	0.572	0.344
	FDM	2.171	1.446	0.935	0.572	0.347
0.5	Sim	2.229	1.548	1.047	0.687	0.446
	CM	2.225	1.558	1.052	0.691	0.446
	FDM	2.226	1.562	1.055	0.695	0.448
0.625	Sim	2.288	1.638	1.150	0.787	0.543
	CM	2.299	1.657	1.163	0.800	0.543
	FDM	2.295	1.652	1.161	0.797	0.542
0.75	Sim	2.339	1.724	1.243	0.883	0.630
	CM	2.367	1.747	1.262	0.898	0.630
	FDM	2.336	1.710	1.223	0.861	0.601

FDM parameters used: minimum stock price=0, maximum stock price=20, minimum volatility=0, maximum volatility=0.5, starting time=0, ending time=0.25, number of steps in the stock price dimension=200, number of steps in the volatility dimension=200, number of steps in the time dimension=400. IT= Ikonen and Toivanen, Sim=Monte Carlo Simulation, FDM=Finite Difference Method as implemented in this dissertation, CM= Chocklingam & Muthuraman.

It is worth to mention here that we could use Monte Carlo simulations to pricing our derivatives. However, it is always good to have another working pricing program as a double check to minimize the pricing error. Another reason is that MC simulations for Heston SVM still have some technical problems. In chapter 5 we will discuss various discretization schemes for Heston model and their failures to capture the real stochastic process are demonstrated. Therefore in chapter 5 when we actually extract the empirical risk aversion through the pricing scheme we use both Monte Carlo simulations and FDM to ensure that we get the same results.

Section V - Market Price of Risk and Volatility Smile

As indicated in the first and second chapters the primary objective of this dissertation is to obtain greater insight into the economic meaning of the market price of risk. Generally speaking, we argue that what is observed as an implied volatility smile is actually a utility centric deviation from the martingale restriction tied to the market price of risk and market risk aversion. The remaining of this chapter explores the relationship between Heston simulated volatility smile and the MPRs.

Volatility smile has been long studied in the literature. Many of studies ascribe the smile to deficiencies in the classical Black-Scholes model, namely, the log-normal asset price movement and the constant volatility assumptions. In particular, people have found that including extra skewness and kurtosis to the underlying movement will reproduce the volatility smile. Similarly, stochastic volatility models (SVM) and jump-diffusion models have been proposed to simulate the smiles. Although people still don't agree on which factor is the most important, "different empirical studies

seem to suggest that this class of models (Pan, Duffie & Singleton's stochastic volatility with correlated jumps model) are best for explaining the option smiles" (Elie Ayache, Wilmott Magazine).

Assuming a fixed set of model parameters we can simulate Heston volatility smiles in two steps: first, we use the method developed in previous sections to get option prices; second, we take these prices from Heston SVM as the 'correct' market prices for Black-Scholes model and extract implied volatilities.²⁰ Figure 3.1 shows a typical volatility smile produced under Heston SVM assuming martingale restriction. Figure 3.2 shows volatility smiles simulated for different time to maturities.

The relationship between market prices of risks and volatility smiles has seldom been explored in the literature. The latest related dissertation by Doran & Ronn (2008) demonstrate by quasi-Monte Carlo simulations the importance of a negative volatility risk premium²¹ in explaining why Black implied volatility is higher than realized volatility in energy markets under stochastic volatility and jump (SVJ) models. In this section we differentiate the impact of market price of asset risk from that of market price of volatility risk directly on the shape of implied volatility smiles. As mentioned earlier, convention assumes martingale restriction (hence the market price of asset risk

²⁰ Renault and Touzi (1996) shows that stochastic volatility European option prices produce the volatility smile for any volatility process uncorrelated with the Brownian motion driving the underlying asset price process.

²¹ Again as we point out in the last section, the volatility risk premium in Doran & Ronn (2008) is in fact a correlation-weighted mixture of asset risk premium and volatility risk premium.

is fixed to the Sharpe ratio) and sets the “volatility risk” to zero.²² Here to better understand the role of asset risk in volatility smile, we first relax the martingale restriction assumption to allow asset risk to vary in a certain range, and then we fix it at the Sharpe ratio and examine how other model parameters would affect the shape of smile given a constant asset risk.

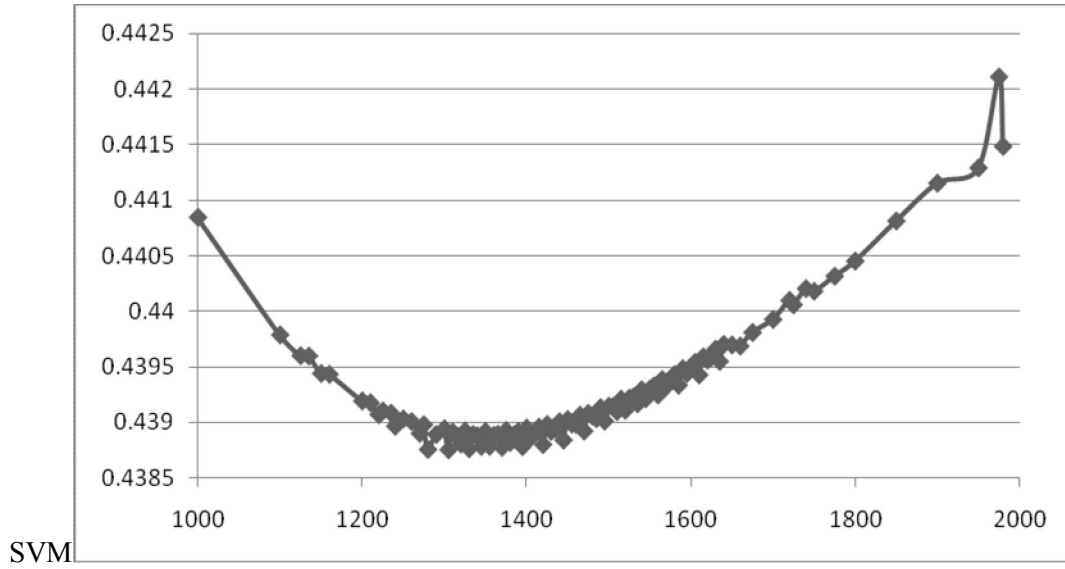


Figure 3 - Simulated Volatility Smile under Heston

A volatility smile simulated under Heston SVM with martingale restriction and zero volatility risk.

²² In fact many researches set the term $\rho\lambda_1 + \sqrt{1-\rho^2}\lambda_2$ to zero. This effectively sets

$$\lambda_2 = -\frac{\rho}{\sqrt{1-\rho^2}} \frac{\mu-r}{\sigma}.$$

drift rate $\mu = 0.05$, interest rate $r = 0.03$, dividend rate $q = 0$, mean reversion rate $\kappa = 1.5$, volatility of volatility $\xi = 0.10$, mean reversion level $\theta = 0.02$, correlation $\rho = -0.02$, initial volatility $v_0 = 0.2$, asset risk $= (\mu - r) / \sqrt{v_0} = 0.0447$, volatility risk = 0.
Current stock price=1300, strike=1300.

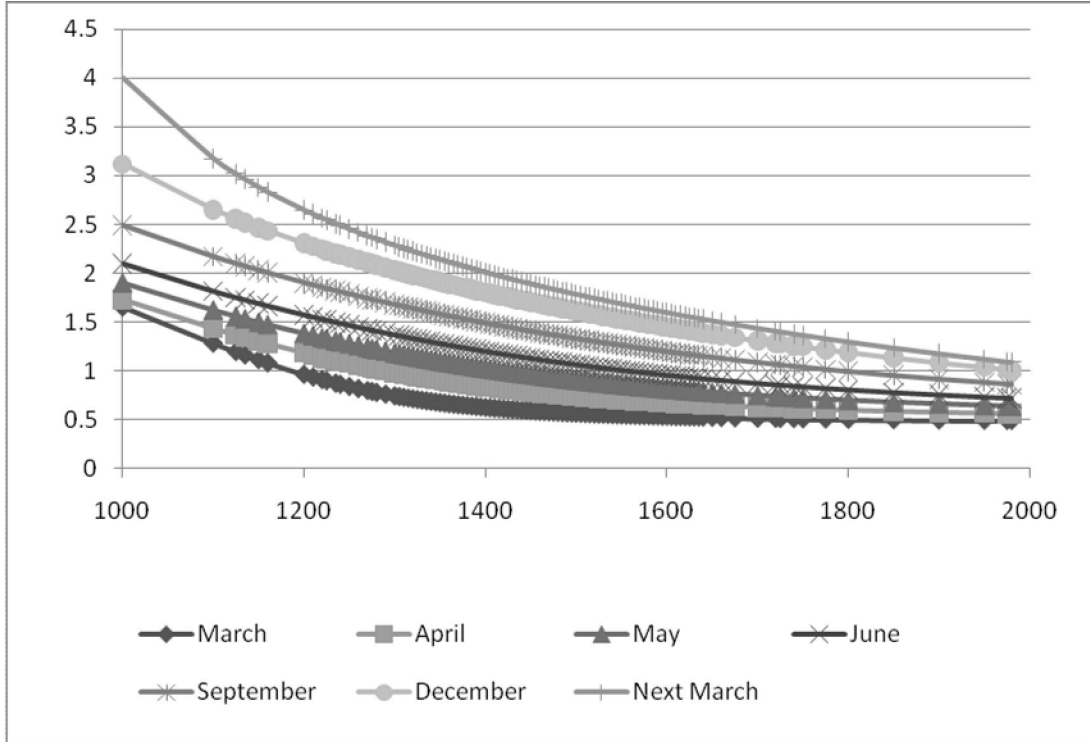


Figure 4 - Simulated Volatility Smiles across Time to Maturities

Figure 4 shows different volatility smiles for different time to maturities. Model parameters are as the following (note that here martingale restriction assumption is relaxed):

drift rate $\mu = 0.05$, interest rate $r = 0.03$, dividend rate $q = 0$, mean reversion rate $\kappa = 1.5$, volatility of volatility $\xi = 0.10$, mean reversion level $\theta = 0.02$, correlation $\rho = -0.02$, initial volatility $v_0 = 0.2$, asset risk = -2, volatility risk = 0.

As first steps of the experiment, Figure 3 and Figure 4 show the sensitivity of European call and put option prices with respect to the two market prices of risks. It is easily noticed that the option prices are more sensitive to the market price of asset risk than to the market price of volatility risk. Notice also that the sensitivity curve for a call is convex while for a put is concave. Out of the curiosity to see whether the type of option has any effect on the sensitivity of option prices with respect to MPRs, we illustrate in Figure 5 that the sensitivity does not change much across European or American options. Hence in this section we mainly consider European options as it is much faster to implement European option pricing on the computer.

Figure 6 illustrates the volatility smiles for European call options simulated from different values of asset risks ranging from -2 to +2 with a common zero volatility risk. Notice the smile shapes are changing with different values of asset risks. It is also observed that the larger the absolute value of the market price of asset risk, the steeper the smile or frown it produces.

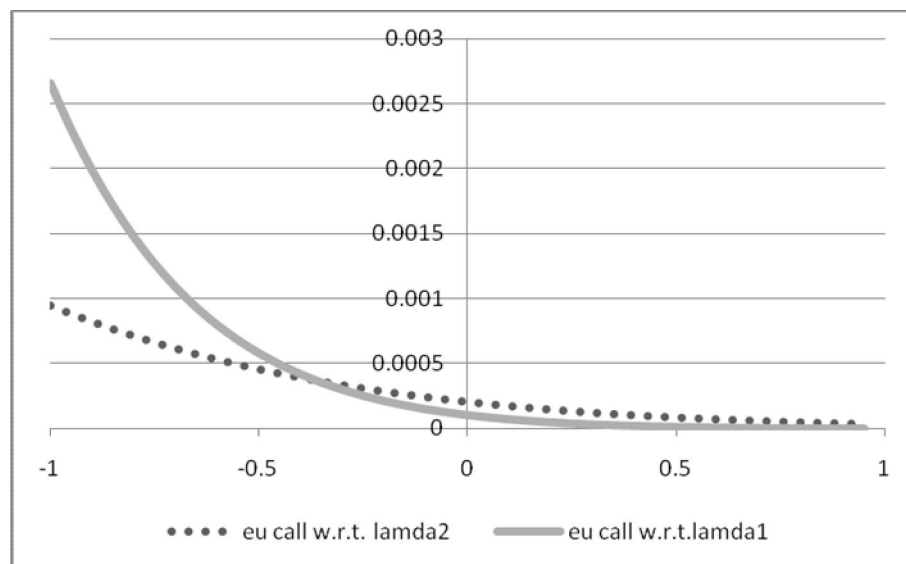


Figure 5 - Sensitivity of European Call Prices with respect to Market Prices of Risks

Figure 5 shows how European call prices change with asset risk and volatility risk. X-axis is market prices of risks, Y-axis is the call price. The line for asset risk is much steeper than that for volatility risk.

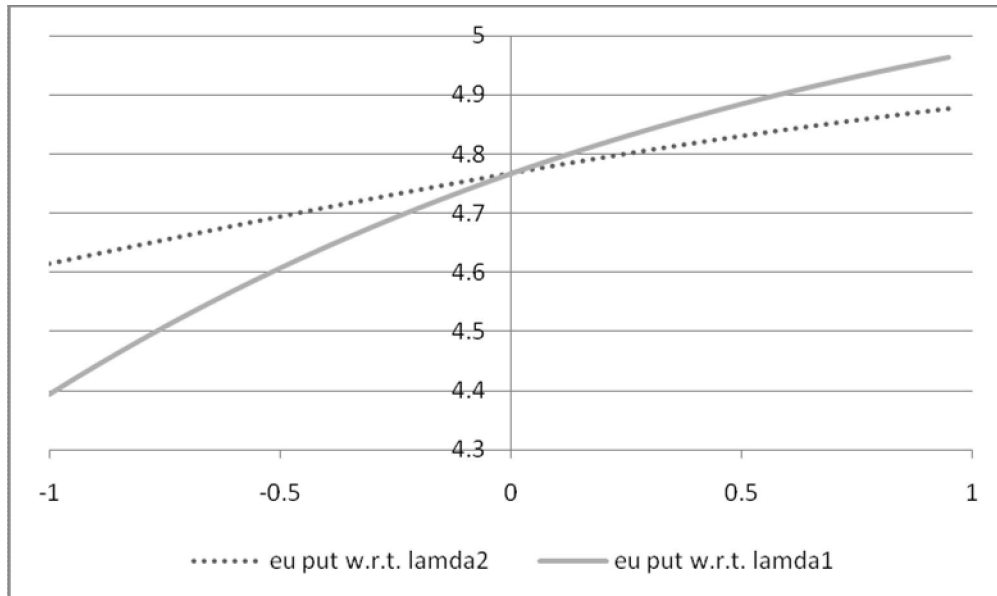


Figure 6 - Sensitivity of European Put Prices with respect to Market Prices of Risks

Figure 6 shows how European put prices change with asset risk and volatility risk. X-axis is market prices of risks, Y-axis is the put price. The line for asset risk is much steeper than that for volatility risk.

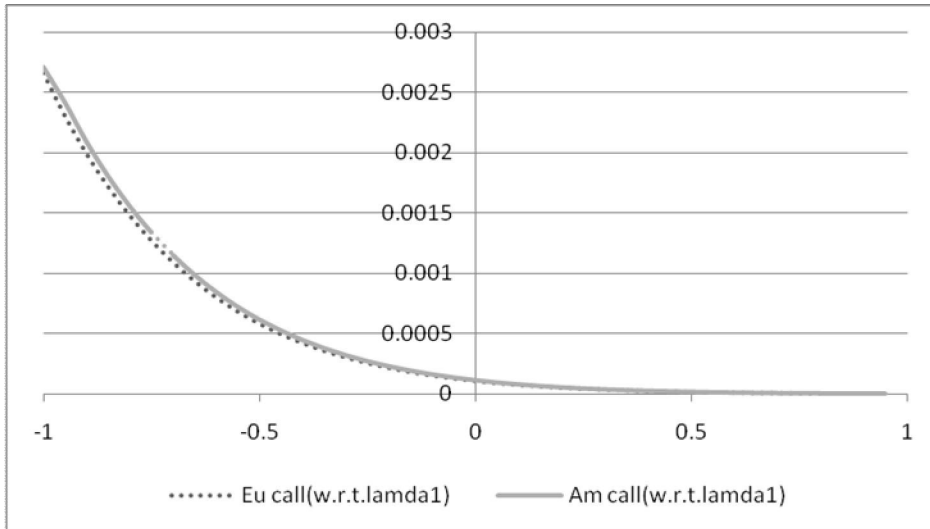


Figure 7 - Sensitivity of European and American Call Prices with respect to Market Price of Asset Risk

Figure 7 shows that there is no much difference in the sensitivity between European and American options. X-axis is market prices of risks, Y-axis is the call price.

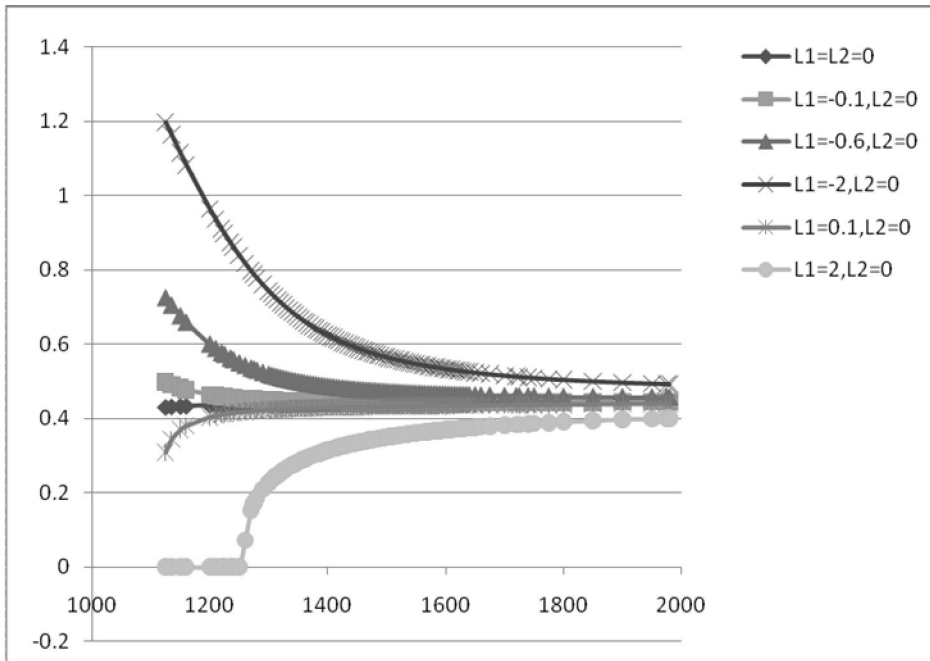


Figure 8 - Simulated Volatility Smiles/Frowns for European Calls with Different Market Prices of Asset Risks

Figure 8 examines the effect of market price of asset risk under a “microscope”: we varied the market price of asset risk from 0.04 to 0.0447 to 0.05. What we observe is that even for a tiny change of the value of asset risk the volatility smiles change significantly. This shows that the market price of asset risk, hence the martingale restriction assumption, is very important in determining the shape of the smile. In addition, it seems from both Figure 7 and Figure 8 that the asset risk mostly affects deeply in-the-money and near the money calls and has minimal effect on deeply out-of-money calls. For comparison, we look at the effects on volatility smiles of other parameters in the model such as the initial volatility, the mean reversion rate, the mean reversion level, volatility and volatility and illustrate them in Figure 9 to Figure 13. It is easily noticed that except for the initial volatility all other parameters do not influence the shape of smiles very much even for a relatively large change of values. Since in equation (3.5) the asset risk is multiplied by the initial volatility, however, it is reasonable to observe that changing initial volatility also impacts the shape significantly.

In summary, these simulated experiments have shown that market price of asset risk is the most important factor that affects the smile shape, other parameters only contributing marginal effects. The correlation coefficient has larger effect on deep out-the-money calls, and the mean reversion rate affects both in and out the money calls. There are almost no difference on smile shapes due to mean reversion level and volatility of volatility. This observation implies that the validity of martingale restriction assumption may be a crucial factor to understand the implied volatility smile problem. The observation is also consistent with current findings that large volatility risk premium is needed to explain the smile.

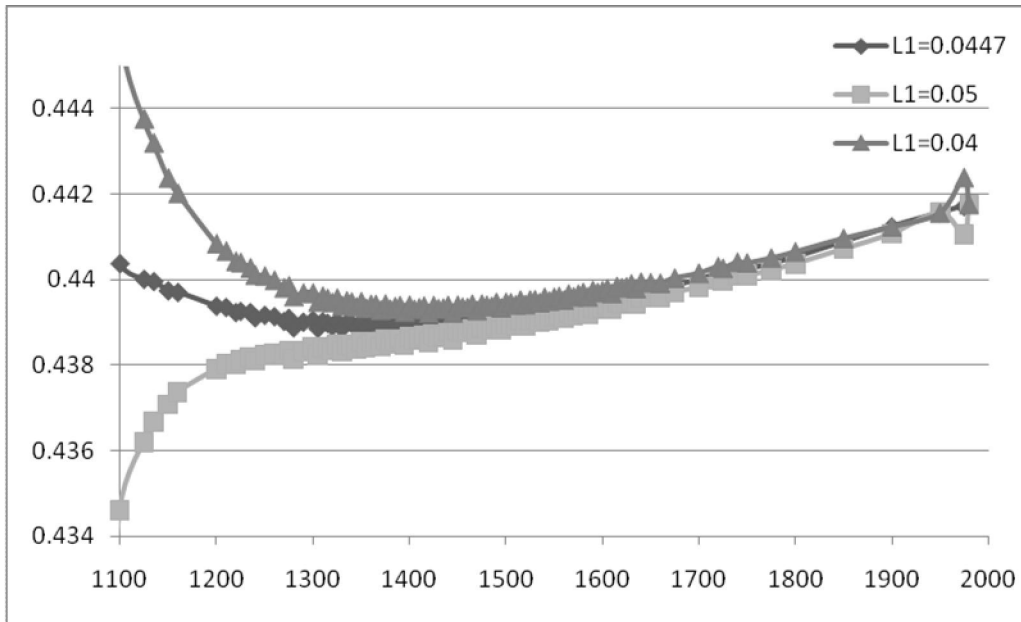


Figure 9 - Simulated Volatility Smiles for European Calls for a Small Range of Market Prices of Asset Risks

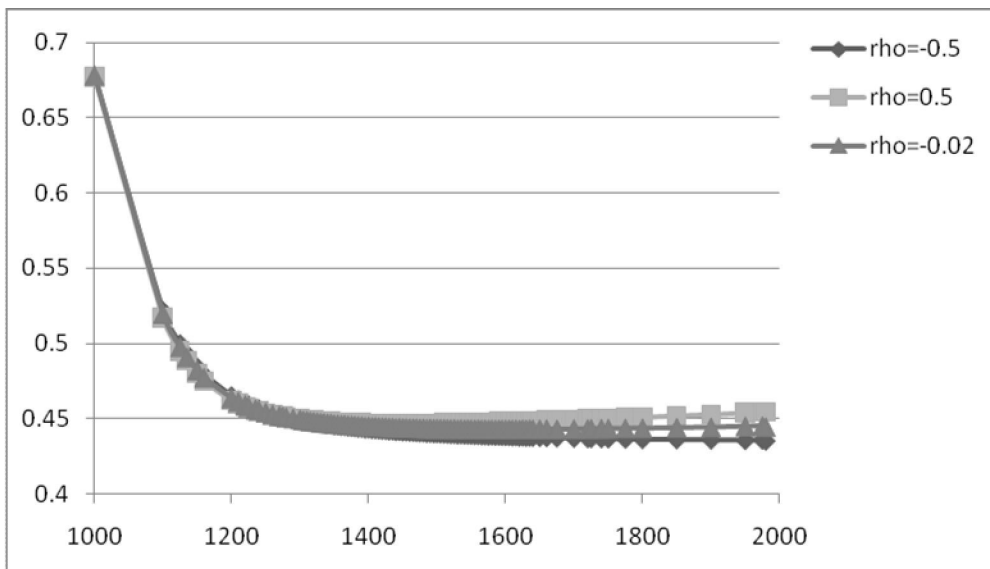


Figure 10 - Simulated Volatility Smiles for European Calls for Different Values of Correlation Coefficients

We vary the correlation coefficient from -0.5 to -0.02 to 0.5 and get insignificant changes of the smiles. Correlation has most impact on deep out-the-money calls.

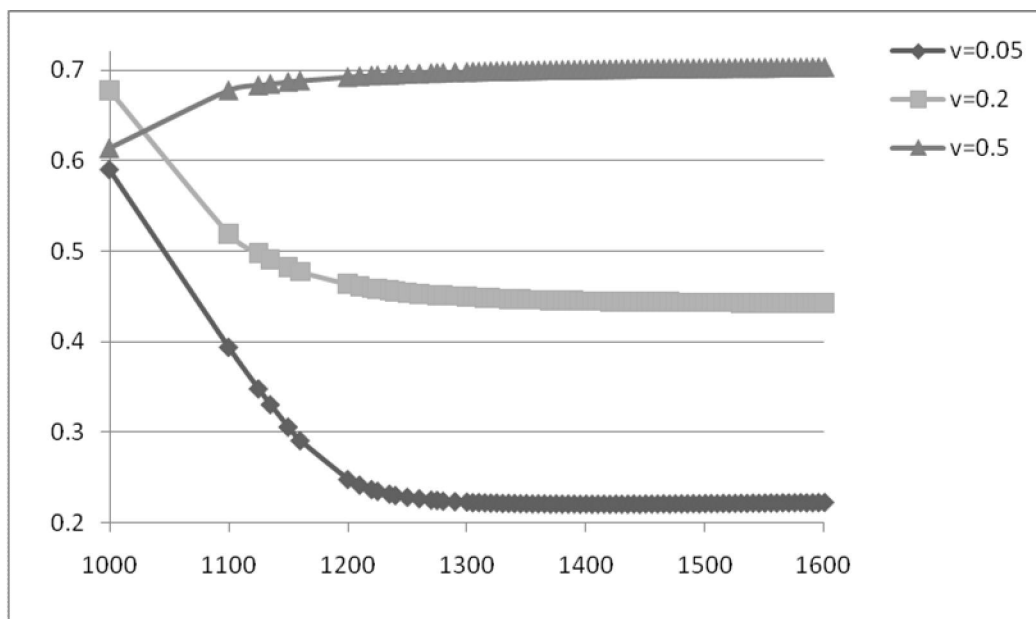


Figure 11 - Simulated Volatility Smile for European Calls with respect to Initial Volatility Level

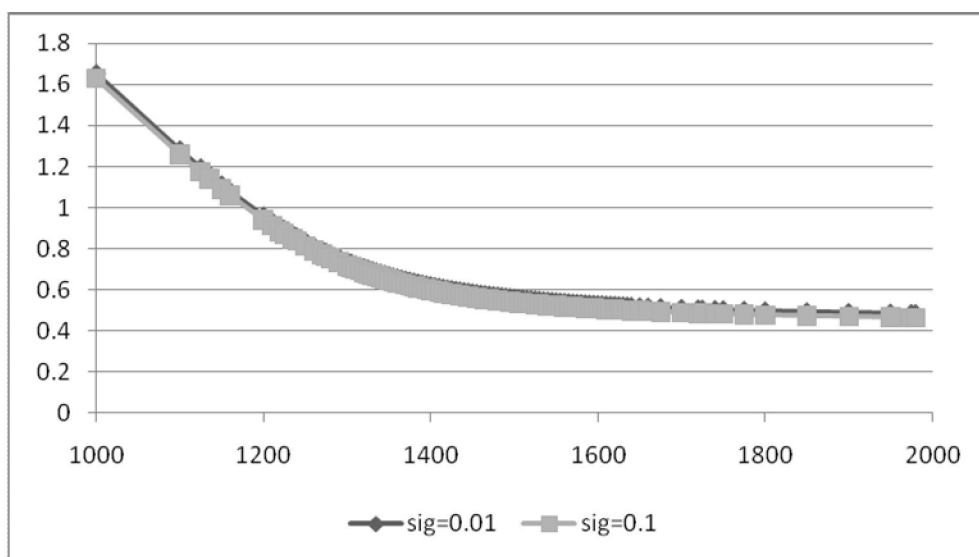


Figure 12 - Simulated Volatility Smile for European Calls with respect to Volatility of Volatility

We vary the volatility of volatility coefficient from 0.01 to 0.1 and observe no visually significant differences in the simulated smiles for all strikes.

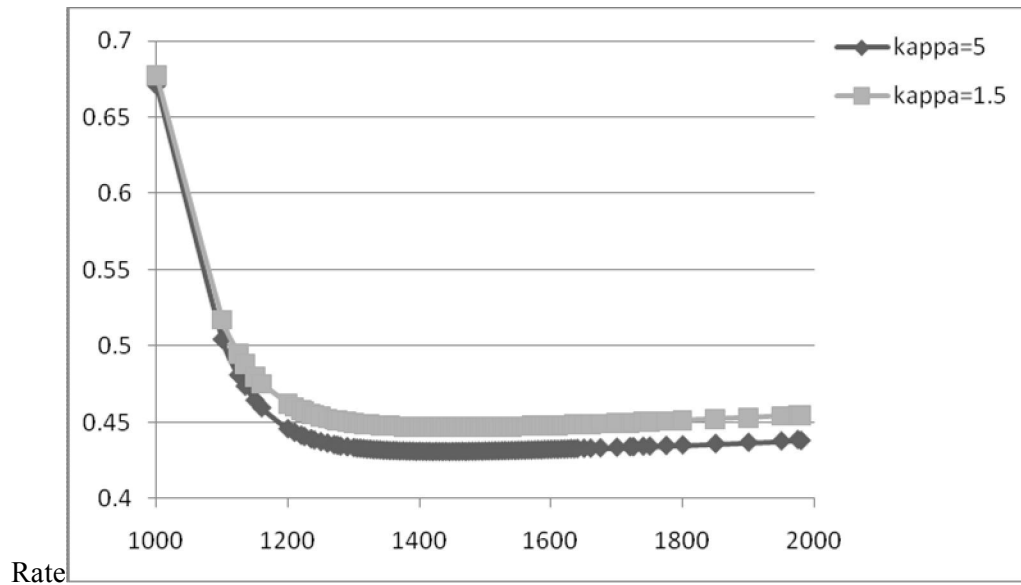


Figure 13 - Simulated Volatility Smile for Calls with respect to Mean Reversion

We vary the mean reversion rate from 1.5 to 5 and find small differences in the simulated smiles across most strikes.

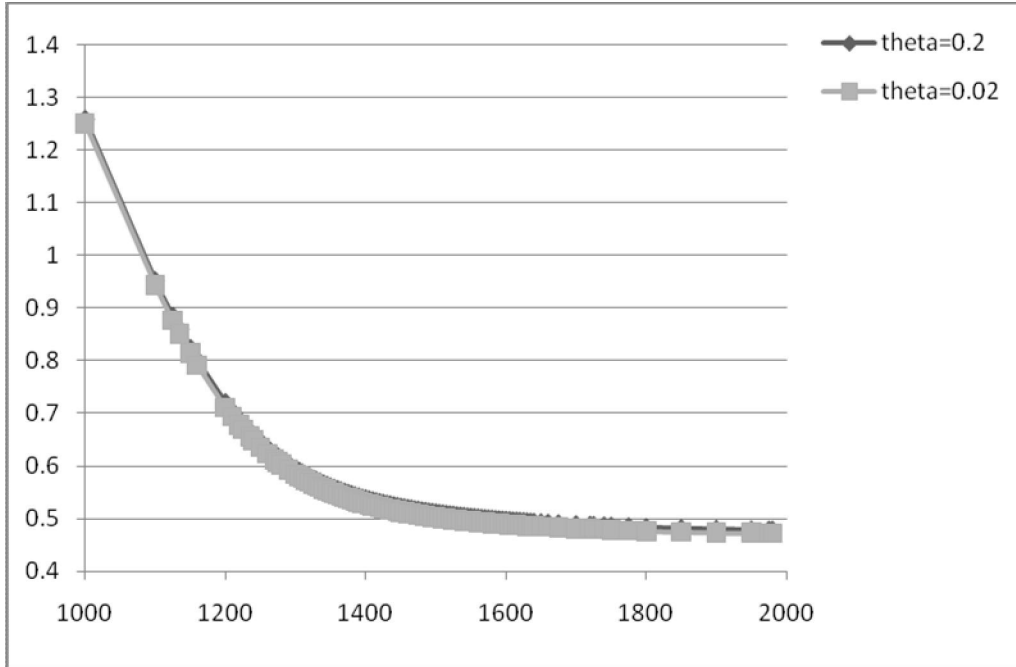


Figure 14 - Simulated Volatility Smile for European Calls with respect to Mean Reversion Level

We vary the mean reversion level from 0.02 to 0.2 and find no significant differences in the simulated smiles for all strikes.

Section VI – Conclusion

In this chapter the Finite Difference Method for option pricing is introduced in detail. We first apply the idea to the one-factor partial differential equation, then extend it to a two-factor case and apply it to the Heston SVM. Borrowing the penalty method from Duffy we successfully price the American option. Numerical comparisons are made with other research and find the algorithms very accurate. With the fully developed code we examine the sensitivity of simulated Heston volatility smile with respect to

various model parameters including the two market prices of risks. We find that the smile is most sensitive to the value of market price of asset risk, the market price of volatility risk, correlation between random sources and the initial volatility level. By the results in chapter 2, this observation implies that the market risk aversion determines the shape of simulated volatility smile, i.e. for each level of market risk aversion the model implied smile is different. The common practice of assuming specific values for these parameters followed by fitting the model to market implied volatility smiles makes no economic sense. In comparison, determining the risk premium and risk aversion by minimizing the distance between market smile and the model implied smile is equivalent to solving an aggregation problem of market risk aversions (hence averaging the market pricing kernels). But in fact the most sensible way is to assume heterogeneous investor groups with heterogeneous pricing kernels associated with each market completed by a put option with certain strike and maturity, then to extract against strikes for each fixed maturity the corresponding market risk aversion. It will be interesting to compare the average MRA with individual MRAs. The hypothesis is that: for a certain range of strikes, the governing market pricing kernel is close to the average of individual pricing kernels (so the individual MRAs are close to the average MRA for that range), while for deep OTM options the governing pricing kernel is those associated with individuals with certain characteristics (so the individual MRAs are very different from the average). As a further step we can examine the MRA surface (cross section and time series) for any pattern. An MRA index may be constructed to monitor the fluctuations of market investors' subjective sensitivity. Finally, we can regress the individual risk aversions with the volumes of traded options to see if liquidity is a factor explaining the differing risk aversions across strikes.

APPENDIX 3.1

VBA FDM ALGORITHMS FOR EUROPEAN OPTIONS

Option Explicit

Created @ 02/20/2009 by Qian Han

Public Sub EuropeanSolver(ByVal callOrPut As String)

Call Initializor

Dim currentTime As Double

Dim c_count As Integer ' column counter

Dim r_count As Integer ' row counter

Dim AcoeffArray() As Double ' dynamic array to store lower diagonal coefficients

Dim BcoeffArray() As Double ' dynamic array to store main diagonal coefficients

Dim CcoeffArray() As Double ' dynamic array to store upper diagonal coefficients

Dim CrosscoeffArray() As Double ' dynamic array to store cross term coefficients

Dim FcoeffArray() As Double ' dynamic array to store forcing term coefficients

Dim GcoeffArray() As Double ' dynamic array to store coefficients that adjusts for the first and last equations

Dim RHSArray() As Double ' dynamic array to store the right hand side vector of the system of equations

Dim tempMatrix() As Double ' to store temporary results/option prices

ReDim tempMatrix(0 To xSteps, 0 To ySteps)

‘ Before we initialize the initial and boundary conditions,

‘ we need to know which type of option we are considering

If callOrPut = "Call" Then ' if this is a call

```

‘ First initialize the tempMatrix by initial conditions
Dim i As Integer
Dim j As Integer
For i = LBound(tempMatrix, 1) To UBound(tempMatrix, 1)
    For j = LBound(tempMatrix, 2) To UBound(tempMatrix, 2)
        tempMatrix(i, j) = MaxFunction(XARR(i), K)
    Next j
Next i

‘Now the boundary conditions...
For j = LBound(tempMatrix, 2) To UBound(tempMatrix, 2)
    ‘Impose the left boundary condition in the stock price dimension ( S --> 0 )
    tempMatrix(0, j) = MaxFunction(XARR(LBound(XARR)), K)
Next j

‘Impose the right boundary condition in the volatility dimension ( V --> +infinity )
‘When volatility goes to infinity we assume the option price just behaves as the
Black-Scholes; Or we can just assume as Heston (1993)
For i = LBound(tempMatrix, 1) To UBound(tempMatrix, 1)
    tempMatrix(i, UBound(tempMatrix, 2)) = BSOptionValue(1, XARR(i), K, r, d,
Tto - Tfrom, Sqr(YARR(UBound(YARR))))
    tempMatrix(i, UBound(tempMatrix, 2)) = XARR(i)
Next i

Else

If callOrPut = "Put" Then ' if this is a put
    ‘First initialize the tempMatrix by initial conditions
    For i = LBound(tempMatrix, 1) To UBound(tempMatrix, 1)
        For j = LBound(tempMatrix, 2) To UBound(tempMatrix, 2)

```

```

        tempMatrix(i, j) = MaxFunction(K, XARR(i))
    Next j
Next i

‘The following boundary conditions follow Duffy. chapter 22, eqn. (22.15) ~
(22.18)

For j = LBound(tempMatrix, 2) To UBound(tempMatrix, 2)
    ‘ Impose the left boundary condition in the stock price dimension ( S --> 0 )
        tempMatrix(0, j) = K
    Next j
    For i = LBound(tempMatrix, 1) To UBound(tempMatrix, 1)
        Impose the left boundary condition in the volatility dimension ( V--> 0 )
        tempMatrix(i, 0) = MaxFunction(K, XARR(i))
    Next i
End If
End If

‘ Set current time
currentTime = TARR(LBound(TARR))

‘ Now we are ready to do time-marching
Dim counter As Integer
Dim temp() As Double
Do Until (currentTime > TARR(UBound(TARR)) - 0.5 * ht)
    For c_count = LBound(YARR) + 1 To UBound(YARR) - 1
        ReDim temp(LBound(XARR) + 1 To UBound(XARR) - 1)

        temp = GetColumn(A1discrete, c_count)
        ReDim AcoeffArray(LBound(temp) + 1 To UBound(temp))
    
```

```

AcoeffArray = GetArrayElements(temp, LBound(temp) + 1, UBound(temp))
ReDim BcoeffArray(LBound(temp) To UBound(temp))
BcoeffArray = GetColumn(B1discrete, c_count)
temp = GetColumn(C1discrete, c_count)
ReDim coeffArray(LBound(temp) To UBound(temp) - 1)
CcoeffArray = GetArrayElements(temp, LBound(temp), UBound(temp) - 1)
ReDim CrosscoeffArray(LBound(temp), UBound(temp))
CrosscoeffArray = GetColumn(Cross1discrete, c_count)
ReDim FcoeffArray(LBound(temp), UBound(temp))
FcoeffArray = GetColumn(F1discrete, c_count)
ReDim GcoeffArray(LBound(temp) To UBound(temp))
GcoeffArray(LBound(GcoeffArray)) = tempMatrix(0, c_count) * A1discrete(1,
c_count) ' Note: using left boundary condition here
GcoeffArray(UBound(GcoeffArray)) = tempMatrix(xSteps, c_count) *
C1discrete(xSteps - 1, c_count) ' Note: using right boundary condition here
ReDim RHSArray(LBound(temp) To UBound(temp))
For counter = LBound(RHSArray) To UBound(RHSArray)
    RHSArray(counter) = (1 / (8 * hx * hy)) * ht * Cross1discrete(counter,
c_count) * (tempMatrix(counter + 1, c_count - 1) _
                + tempMatrix(counter - 1, c_count + 1) -
tempMatrix(counter - 1, c_count - 1) - tempMatrix(counter + 1, c_count + 1)) _
                - 1 * tempMatrix(counter, c_count) + ht *
FcoeffArray(counter) - GcoeffArray(counter)
Next counter
temp = LUdecomposition(AcoeffArray, BcoeffArray, CcoeffArray, RHSArray)
Transfer result to the tempMatrix

```

```

For i = LBound(temp) To UBound(temp)
    tempMatrix(i, c_count) = temp(i)
Next i
Next c_count ' End of j-loop
currentTime = currentTime + ht / 2 ' Move to the next time level
For r_count = (LBound(XARR) + 1) To (UBound(XARR) - 1) ' start from the
second row
    ReDim temp(LBound(YARR) + 1 To UBound(YARR) - 1)
    temp = GetRow(A2discrete, r_count)
    ReDim AcoeffArray(LBound(temp) + 1 To UBound(temp))
    AcoeffArray = GetArrayElements(temp, LBound(temp) + 1, UBound(temp))
    ReDim BcoeffArray(LBound(temp) To UBound(temp))
    BcoeffArray = GetRow(B2discrete, r_count)
    temp = GetRow(C2discrete, r_count)
    ReDim CcoeffArray(LBound(temp) To UBound(temp) - 1)
    CcoeffArray = GetArrayElements(temp, LBound(temp), UBound(temp) - 1)
    ReDim CrosscoeffArray(LBound(temp) To UBound(temp))
    CrosscoeffArray = GetRow(Cross2discrete, r_count)
    ReDim FcoeffArray(LBound(temp) To UBound(temp))
    FcoeffArray = GetRow(F2discrete, r_count)
    ReDim GcoeffArray(LBound(temp) To UBound(temp))
    GcoeffArray(LBound(GcoeffArray)) = tempMatrix(r_count, LBound(YARR))
* A2discrete(r_count, 1) ' Note: using left boundary condition here
    GcoeffArray(UBound(GcoeffArray)) = tempMatrix(r_count, UBound(YARR))
* C2discrete(r_count, ySteps - 1) ' Note: using right boundary condition here
    ReDim RHSArray(LBound(temp) To UBound(temp))

```



```

For counter = LBound(RHSArray) To UBound(RHSArray) ' loop through
column

    RHSArray(counter) = (1 / (8 * hx * hy)) * ht * Cross2discrete(r_count,
counter) * (tempMatrix(r_count - 1, counter + 1) _
            + tempMatrix(r_count + 1, counter - 1) -
tempMatrix(r_count - 1, counter - 1) - tempMatrix(r_count + 1, counter + 1)) _
            - 1 * tempMatrix(r_count, counter) + ht *
FcoeffArray(counter) - GcoeffArray(counter)

Next counter

temp = LUDecomposition(AcoeffArray, BcoeffArray, CcoeffArray, RHSArray)
For i = LBound(temp) To UBound(temp)
    tempMatrix(r_count, i) = temp(i)
Next i

Next r_count ' End of i-loop

currentTime = currentTime + ht / 2 ' Move to the next time level

Loop ' Do-Until Loop

If callOrPut = "Call" Then ' if this is a call
    For j = LBound(tempMatrix, 2) To UBound(tempMatrix, 2)
        tempMatrix(UBound(tempMatrix, 1), j) = tempMatrix(UBound(tempMatrix, 1)
- 1, j) + hx * Exp(-(r - d) * (Tto - Tfrom))
    Next j

    Dim norm1 As Double
    Dim norm2 As Double

    For i = 1 To xSteps - 1
        norm1 = ht * (r - d) * XARR(i) / hx
        norm2 = kappa * theta * ht / hy

```

```

tempMatrix(i, 0) = (1 - norm1 - norm2 - r * ht) * tempMatrix(i, 0) + norm1 *
tempMatrix(i + 1, 0) + norm2 * tempMatrix(i, 1)

```

```

Next

```

‘ Alternatively, we can use implicit scheme for the leg one and explicit scheme for leg two

```

Do Until (currentTime > TARR(UBound(TARR)) - 0.5 * ht)

```

```

    Dim a() As Double

```

```

    Dim b() As Double

```

```

    Dim c() As Double

```

```

    ReDim a(1 To xSteps - 1)

```

```

    ReDim b(1 To xSteps - 1)

```

```

    ReDim c(1 To xSteps - 1)

```

```

    Dim ii As Integer

```

```

    For ii = 1 To xSteps - 1

```

```

        a(ii) = -0.5 * ht * mu * XARR(ii) / hx

```

```

        b(ii) = -1 - (r - d) * ht

```

```

        c(ii) = 0.5 * ht * (r - d) * XARR(ii) / hx

```

```

    Next ii

```

```

    Dim a_tr() As Double

```

```

    Dim c_tr() As Double

```

```

    ReDim a_tr(2 To xSteps - 1)

```

```

    a_tr = GetArrayElements(a, 2, xSteps - 1)

```

```

    ReDim c_tr(1 To xSteps - 2)

```

```

    c_tr = GetArrayElements(c, 1, xSteps - 2)

```

```

    Dim rhs() As Double

```

```

    ReDim rhs(1 To xSteps - 1)

```

```

    rhs(1) = -1 * tempMatrix(1, 0) - a(1) * tempMatrix(0, 0)
    For ii = 1 To xSteps - 1
        rhs(ii) = -1 * tempMatrix(ii, 0)
    Next
    rhs(xSteps - 1) = -1 * tempMatrix(xSteps - 1, 0) - c(xSteps - 1) *
tempMatrix(xSteps, 0)
    Dim answer() As Double
    ReDim answer(1 To xSteps - 1)
    answer = LUdecomposition(a_tr, b, c_tr, rhs)
    For ii = 1 To xSteps - 1
        tempMatrix(ii, 0) = answer(ii)
    Next
    currentTime = currentTime + ht / 2
    For ii = 1 To xSteps - 1
        tempMatrix(ii, 0) = tempMatrix(ii, 0) + (kappa * theta - sigma * (rho *
lamda1 + Sqr(1 - rho * rho) * lamda2)) * ht * (tempMatrix(ii, 1) - tempMatrix(ii, 0)) /
hy
    Next ii
    currentTime = currentTime + ht / 2
Loop
Else
    If callOrPut = "Put" Then ' if this is a put
        For j = LBound(tempMatrix, 2) To UBound(tempMatrix, 2)
            tempMatrix(UBound(tempMatrix, 1), j) = tempMatrix(UBound(tempMatrix,
1) - 1, j)
        Next j

```

```

        For i = LBound(tempMatrix, 1) To UBound(tempMatrix, 1)
            tempMatrix(i, UBound(tempMatrix, 2)) = tempMatrix(i,
UBound(tempMatrix, 2) - 1)
        Next i
    End If
End If

ThisWorkbook.Worksheets("prices").Cells.ClearContents
Dim myRange As Range
Set myRange = ThisWorkbook.Worksheets("prices").Range("A1")
For i = LBound(tempMatrix, 1) To UBound(tempMatrix, 1)
    For j = LBound(tempMatrix, 2) To UBound(tempMatrix, 2)
        myRange.Offset(i * (UBound(tempMatrix, 2) - LBound(tempMatrix, 2) + 1) +
j + 1, 0).Value = XARR(i)
        myRange.Offset(i * (UBound(tempMatrix, 2) - LBound(tempMatrix, 2) + 1) +
j + 1, 1).Value = YARR(j)
        myRange.Offset(i * (UBound(tempMatrix, 2) - LBound(tempMatrix, 2) + 1) +
j + 1, 2).Value = tempMatrix(i, j)
    Next j
Next i

Dim numPoints As Long
numPoints = (UBound(XARR) + 1) * (UBound(YARR) + 1)
Dim stockArray() As Double
ReDim stockArray(1 To numPoints)
Dim volArray() As Double
ReDim volArray(1 To numPoints)
Dim priceArray() As Double

```

```

ReDim priceArray(1 To numPoints)

Dim Ones() As Double

ReDim Ones(1 To numPoints)

For i = 1 To numPoints
    Ones(i) = 1
Next i

Dim independentVariable() As Double

ReDim independentVariable(1 To numPoints, 1 To 3)

Dim OLSresult() As Variant

ReDim OLSresult(1 To 3) As Variant

For i = LBound(tempMatrix, 1) To UBound(tempMatrix, 1)
    For j = LBound(tempMatrix, 2) To UBound(tempMatrix, 2)
        stockArray(i * (UBound(tempMatrix, 2) - LBound(tempMatrix, 2) + 1) + j + 1)
= XARR(i)
        volArray(i * (UBound(tempMatrix, 2) - LBound(tempMatrix, 2) + 1) + j + 1) =
YARR(j)
        priceArray(i * (UBound(tempMatrix, 2) - LBound(tempMatrix, 2) + 1) + j + 1)
= tempMatrix(i, j)
    Next j
Next i

For i = 1 To numPoints
    independentVariable(i, 1) = Ones(i)
    independentVariable(i, 2) = stockArray(i)
    independentVariable(i, 3) = volArray(i)
Next i

OLSresult = OLSregress(priceArray, independentVariable)

```

```

optionPrice = Exp(OLSresult(1) + OLSresult(2) * S + OLSresult(3) * v)
Dim returnedValue As Double
optionPrice = VBInterp3D(stockArray(0), volArray(0), priceArray(0), numPoints, S, v,
returnedValue, "P", "P")
optionPrice = Interpolate2DArray(tempMatrix, XARR, YARR, S, v)
End Sub ' EuropeanSolver()

```

APPENDIX 3.2

ALGORITHM FOR INITIALIZOR

Option Explicit

' Declare all global variables

' Parameters

Public r As Double ' Interest Rate

Public d As Double ' Dividend Rate

Public mu As Double ' Drift Rate for the underlying stock process

Public K As Double ' Strike Price

Public S As Double ' Spot Price

Public v As Double ' Initial Volatility

Public sigma As Double ' Volatility of Voatility

Public kappa As Double ' Mean Reversion Rate

Public theta As Double ' Mean Reversion Level

Public rho As Double ' Correlation Coefficient between Two Brownian Motions

Public lamda1 As Double ' Market Price of Asset Risk

Public lamda2 As Double ' Market Price of Volatility Risk

Public Xfrom As Double ' Minimum Value of stock price

Public Xto As Double ' Maximum Value of stock price

Public Yfrom As Double ' Minimum Value of volatility

Public Yto As Double ' Maximum Value of volatility

Public Tfrom As Double ' Minimum Value of time

Public Tto As Double ' Maximum Value of time

```

Public xSteps As Double ' Number of intervals in stock price dimension
Public ySteps As Double ' Number of intervals in volatility dimension
Public tSteps As Double ' Number of intervals in time dimension
Public hx As Double    ' Mesh size of stock price dimension
Public hy As Double    ' Mesh size of volatility dimension
Public ht As Double    ' Mesh size of time dimension
Public XARR() As Variant ' Array of discretized stock prices
Public YARR() As Double ' Array of discretized volatilities
Public TARR() As Double ' Array of discretized time levels
' The following two variables are used in American option pricing
Public EPS As Double ' the epsilon in the penalty term
Public c As Double ' the C variable in the  $f(P)$  penalty term
' Since at each fixed time level, the coefficient matrices in the PDE don't change
' we initialize them once in the 'initializor' to improve efficiency
' Note: they are matrices in the SVM compared with being vectors in the standard
' BS PDE because we break the PDE into two legs using splitting scheme: for each
' fixed level of stock price( or volatility level) there exists a vector of coefficients.
Public A1discrete() As Double ' Lower Diagonal Matrix for the first leg PDE
Public A2discrete() As Double ' Lower Diagonal Matrix for the second leg PDE
Public B1discrete() As Double ' Main Diagonal Matrix for the first leg PDE
Public B2discrete() As Double ' Main Diagonal Matrix for the second leg PDE
Public C1discrete() As Double ' Upper Diagonal Matrix for the first leg PDE
Public C2discrete() As Double ' Upper Diagonal Matrix for the second leg PDE
Public Cross1discrete() As Double ' Cross Term Matrix for the first leg PDE
Public Cross2discrete() As Double ' Cross Term Matrix for the second leg PDE
Public F1discrete() As Double ' RHS Matrix for the first leg PDE (Forcing Term)

```



```

Public F2discrete() As Double ' RHS Matrix for the second leg PDE (Forcing Term)

Public optionPrice As Double ' Store the calculated option price

Public numStrikes As Double ' Number of strikes in the data

Public numMaturities As Double ' Number of maturity dates in the data

Public numOfObs As Double ' Cross-section number of observations

Public Sub Initializor()

' Here we initialize all global parameters based upon user's input
With ThisWorkbook.Worksheets("parameters")

    S = .Range("spotprice").Value
    r = .Range("interestrate").Value
    d = .Range("dividendrate").Value
    d = r ' If this is a futures option the dividend rate is the same as interest rate
    kappa = .Range("meanreversionrate").Value
    theta = .Range("meanreversionlevel").Value
    sigma = .Range("volofvol").Value
    mu = .Range("driftrate").Value
    rho = .Range("corr").Value
    v = .Range("initialvol").Value
    Xfrom = .Range("minStock").Value
    Xto = .Range("maxStock").Value
    Yfrom = .Range("minVol").Value
    Yto = .Range("maxVol").Value
    Tfrom = 0 ' Always start from 0
    xSteps = .Range("xSteps").Value
    ySteps = .Range("ySteps").Value
    tSteps = .Range("tSteps").Value

```

```

End With

' Set the mesh sizes in all three dimensions

hx = (Xto - Xfrom) / xSteps
hy = (Yto - Yfrom) / ySteps
ht = (Tto - Tfrom) / tSteps

' Check the mean reverting parameters

' To guarantee that the volatility cannot reach zero

' Condition:  $2 * \kappa * \theta > \sigma^2$ 

If 2 * kappa * theta <= sigma * sigma Then

    MsgBox "Mean reverting process parameters are invalid! "

Exit Sub

End If

' Apply the constraint condition as in Duffy, eqn. (22.32) to ensure stability

Do Until ht <= 1 / (mu * Xto / hx + (kappa * theta) / hy + r)

    tSteps = tSteps * 2

    ht = (Tto - Tfrom) / tSteps

Loop

'Update the value in cell "tSteps"

ThisWorkbook.Worksheets("parameters").Range("tSteps").Value = tSteps

ReDim XARR(0 To xSteps)

ReDim YARR(0 To ySteps)

ReDim TARR(0 To tSteps)

'Set the mesh points in three dimensions

XARR(0) = Xfrom

Dim i As Integer

For i = 1 To xSteps

```

```

        XARR(i) = XARR(i - 1) + hx
    Next
    YARR(0) = Yfrom
    Dim j As Integer
    For j = 1 To ySteps
        YARR(j) = YARR(j - 1) + hy
    Next
    TARR(0) = Tfrom
    For j = 1 To tSteps
        TARR(j) = TARR(j - 1) + ht
    Next
    'Initialize the coefficient matrices in the PDE
    'Note: diffusion, convection terms are HARD CODED
    Dim temp1 As Double ' to store convection term
    Dim temp2 As Double ' to store diffusion term
    Dim temp3 As Double ' to store zero term
    Dim temp4 As Double ' to store cross term
    ReDim A1discrete(1 To xSteps - 1, 1 To ySteps - 1)
    ReDim B1discrete(1 To xSteps - 1, 1 To ySteps - 1)
    ReDim C1discrete(1 To xSteps - 1, 1 To ySteps - 1)
    ReDim Cross1discrete(1 To xSteps - 1, 1 To ySteps - 1)
    ReDim F1discrete(1 To xSteps - 1, 1 To ySteps - 1)
    ReDim A2discrete(1 To xSteps - 1, 1 To ySteps - 1)
    ReDim B2discrete(1 To xSteps - 1, 1 To ySteps - 1)
    ReDim C2discrete(1 To xSteps - 1, 1 To ySteps - 1)
    ReDim Cross2discrete(1 To xSteps - 1, 1 To ySteps - 1)

```

```

ReDim F2discrete(1 To xSteps - 1, 1 To ySteps - 1)
'Coefficient Matrices in leg one
For j = LBound(A1discrete, 2) To UBound(A1discrete, 2)
    For i = LBound(A1discrete, 1) To UBound(A1discrete, 1)
        temp1 = (mu - d - lamda1 * Sqr(YARR(j))) * XARR(i) ' convection term in the
leg-1 PDE
        temp2 = 0.5 * YARR(j) * XARR(i) * XARR(i) ' diffusion term in the leg-1
PDE
        temp3 = -(r - d) ' zero term in the leg-1 PDE
        temp4 = rho * sigma * XARR(i) * YARR(j) ' cross term in the leg-1 PDE
        A1discrete(i, j) = ht * ExpoFittingFactor(hx, temp1, temp2) / (hx * hx) - 0.5 *
temp1 * ht / hx
        B1discrete(i, j) = -1 - 2 * ht * ExpoFittingFactor(hx, temp1, temp2) / (hx * hx)
+ ht * temp3
        C1discrete(i, j) = ht * ExpoFittingFactor(hx, temp1, temp2) / (hx * hx) + 0.5 *
temp1 * ht / hx
        Cross1discrete(i, j) = temp4
        F1discrete(i, j) = 0
    Next
Next
'Coefficient Matrices in leg two
For i = 1 To xSteps - 1
    For j = 1 To ySteps - 1
        temp1 = kappa * (theta - YARR(j)) - sigma * (rho * lamda1 + Sqr(1 - rho * rho)
* lamda2) ' convection term in the leg-2 PDE
        temp2 = 0.5 * YARR(j) * sigma * sigma ' diffusion term in the leg-2 PDE

```

```

temp4 = rho * sigma * XARR(i) * YARR(j) ' cross term in the leg-2 PDE
A2discrete(i, j) = ht * ExpoFittingFactor(hy, temp1, temp2) / (hy * hy) - 0.5 *
temp1 * ht / hy
B2discrete(i, j) = -1 - 2 * ht * ExpoFittingFactor(hy, temp1, temp2) / (hy * hy)
C2discrete(i, j) = ht * ExpoFittingFactor(hy, temp1, temp2) / (hy * hy) + 0.5 *
temp1 * ht / hy
Cross2discrete(i, j) = temp4
F2discrete(i, j) = 0
Next j
Next i
End Sub ' Initializor()

```

APPENDIX 3.3

ALGORITHM FOR LUMATRIX SOLVER

Option Explicit

Option Base 1

'Calculate the betas in the L-matrix and gammas in the U-matrix based on tridiagonal arrays in the A-matrix

Public Function LUdecomposition(ByVal arrayA As Variant, ByVal arrayB As Variant, ByVal arrayC As Variant, ByVal arrayRHS As Variant)

Dim sizeA As Integer

Dim sizeB As Integer

Dim sizeC As Integer

Dim sizeD As Integer

sizeA = UBound(arrayA) ' arrayA runs from 2 to xSteps-1

sizeB = UBound(arrayB) ' arrayB runs from 1 to xSteps-1

sizeC = UBound(arrayC) ' arrayC runs from 1 to xSteps-2

sizeD = UBound(arrayRHS) ' arrayRHS from 1 to xSteps-1

If LBound(arrayRHS) <> 1 Or LBound(arrayB) <> 1 Or LBound(arrayC) <> 1 Then

 MsgBox "Invalid indices! "

 Exit Function

End If

If LBound(arrayA) <> 2 Then

 MsgBox "Invalid indices! "

 Exit Function

```

End If

If UBound(arrayRHS) <> UBound(arrayB) Or UBound(arrayA) <>
UBound(arrayRHS) Then

    MsgBox "Invalid indices! "

    Exit Function

End If

If UBound(arrayC) <> UBound(arrayRHS) - 1 Then

    MsgBox "Invalid indices!"

    Exit Function

End If

' ----- check diagonal dominance

If Abs(arrayB(1)) < Abs(arrayC(1)) Then

    MsgBox "diagonal dominance doesn't hold !"

    Exit Function

End If

If Abs(arrayB(sizeB)) < Abs(arrayA(sizeA)) Then

    MsgBox "diagonal dominance doesn't hold !"

    Exit Function

End If

Dim j As Integer

For j = 2 To sizeC

    If Abs(arrayB(j)) < Abs(arrayA(j)) + Abs(arrayC(j)) Then

        MsgBox "diagonal dominance doesn't hold !"

        Exit Function

    End If

Next j

```

```

'Initialization of beta and gamma arrays

Dim beta() As Double ' beta has the same size and index as arrayB
ReDim beta(1 To sizeB)

Dim gamma() As Double ' gamma has the same size and index as arrayC
ReDim gamma(1 To sizeC)

beta(1) = arrayB(1)
gamma(1) = arrayC(1) / beta(1)

' Elements from 2 to next-to-last

Dim b As Integer

For b = 2 To sizeC      ' note: sizeB - sizeC = 1
    beta(b) = arrayB(b) - (arrayA(b) * gamma(b - 1))
    gamma(b) = arrayC(b) / beta(b)
Next b

' Last element

beta(sizeB) = arrayB(sizeB) - (arrayA(sizeB) * gamma(sizeC))

Dim z() As Double
ReDim z(1 To sizeB)

Dim u() As Double
ReDim u(1 To sizeB)

' Calculate the z's

z(1) = arrayRHS(1) / beta(1)

Dim zz As Integer

For zz = 2 To sizeB
    z(zz) = (arrayRHS(zz) - arrayA(zz) * z(zz - 1)) / beta(zz)
Next

' Then calculate the u's

```



```

u(sizeB) = z(sizeB)
Dim i As Integer
For i = sizeB - 1 To 1 Step -1
    u(i) = z(i) - gamma(i) * u(i + 1)
Next i
LUDecomposition = u
End Function ' LUDecomposition()

```

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CHAPTER 4

EFFICIENT METHOD OF MOMENTS

Previous chapters have considered a pure exchange economy where asset prices are driven by a complete stochastic volatility model and the representative agent has habit formation utility. We also derived the quantitative relation between market price of risk and risk aversion, and described the numerical methods we use to price both European and American options. We demonstrate at least visually that the market prices of risks, the correlation and the initial volatility, all of which key parameters in the formula of market risk aversion (3.16), have the most impact on the shape of volatility smile. This reveals close relationship between MRA and the smile.

Combined with the notion in chapter 2 that each put option with a certain level of strike and time to maturity completes the market, hence corresponding to a representative agent and a pricing kernel, we have the following hypothesis inspired by Cvitanic et al. (2009). There are as many individual market pricing kernels (hence many individual market risk aversions (IMRA)) as the number of strikes. When practitioners commonly cross-fit the model-implied volatility smile to the market smile, they are actually aggregating these pricing kernels and risk aversions. We call the resulting risk aversion “average MRA” (AMRA). The relations of market pricing kernels between the heterogeneous economy and the homogeneous economy in Cvitanic et al. (2009), however, imply that in our options market the market pricing kernel should exhibit average behavior for a certain range of strikes and be dominated by individuals outside this range. We call this the “theoretical MRA” (TMRA).

Considering all these above, we decide to extract the risk aversion under risk

neutrality assumption using three approaches. First, we extract the volatility risk for each strike price, which gives IMRAs for each strike. Second, we extract the volatility risk by minimizing the mean squared error (RMSE) between the theoretical option prices and the market observed prices for all strikes (cross-fitting), which gives us AMRA. Third, we extract the volatility risk by minimizing the RMSE for strikes between (+/- 10% of ATM strike = spot), which produces MRA in the middle range; then we extract the volatility risk for each strike beyond the range to get MRA in the tails. This effectively gives us the TMRA. It will be interesting to compare these MRAs.

Another application of the theoretical results presented in chapter 2 is a risk neutrality test by using the pricing partial differential equation (3.6). We can minimize the RMSE by varying all model parameters such as the mean reversion rate, mean reversion level, the correlation coefficient and the market prices of risks. But we will leave this test to the next chapter. So our major task in this chapter is to extract market price of risk and risk aversion from real market data while fixing other model parameters. We will do this in two steps: first, we estimate all model parameters except the market prices of risks; second, we extract the market prices of risks by minimizing the disparity between model-predicted prices and market observed prices. But how do we get those model parameters such as the drift rate, the mean reversion rate, mean reversion level and correlation etc.?

Various parameter estimation methods for continuous time stochastic volatility models have been developed in the past. Two broad categories can be drawn. One is the so-called cross sectional fitting approach. It allows the user to group all unknown parameters and extract them all at once by minimizing the distance between

theoretical prices and the real prices via either global algorithms such as stochastic annealing, differential evolution, or local minimization algorithms such as levenberg-marquardt (LM) and Nelder-Mead (NM). The advantage for this approach is that it is very easy to program. The disadvantage is that it may introduce inconsistencies with the model and the computing cost is often very high. However, cross-sectional fitting is a natural approach for us to extract parameters in the unrestricted model in chapter 5. The other approach uses time series data (for example, Simulated Method of Moments (SMM), Monte Carlo Markov Chain (MCMC), EMM, General Method of Moments (GMM) etc, see Chernov & Ghysels (1999) for references). In comparison, the main advantage of this group of methods is that they allow users to exploit the dynamics of the underlying process in estimating parameters while the major disadvantage is of course increasing computational cost. As can be quickly concluded this time-series estimation is well suited for our purpose here to estimate those model parameters except the market prices of risks. In particular, we choose the EMM to estimate those model parameters.

This chapter is organized as follows. In the first part we introduce the auxiliary model used by EMM as the score generator function called Semi-Nonparametric (SNP) density. Fitted SNP density for the S&P 500 index data is presented. Section two describes the EMM developed by Gallant & Tauchen (1996) and the reprojection method. In view of the deficiencies of discretization schemes used in EMM, Section Three introduces quadratic –exponential martingale (QEM) corrected scheme by Andersen (2006). Section Four presents the EMM estimated model parameters using Heston SVM. Section Five concludes.

Section I - Semi-Nonparametric (SNP) Density and Fitted Density for Data

4.1.1-SNP

Any asset pricing model such as SVM is in its nature a dynamic system of the state variables and also a data generating process. The transition density is implicitly determined within the model or system. So let us consider a stationary and ergodic multivariate time series $\{y_t\}$ which we are interested in, where each y_t is a vector of length M . Assume that y_t is Markovian, then the stationary density determined by the model can be written as $p(x, y|\rho)$ where x denotes the lagged state vector, ρ the model parameters. SNP is based on the idea of a Hermite expansion on the square root of the conditional density $p(y|x, \rho)$, which leads to a truncated transition density

$f_K(y_t|x)$ that has the form of a location-scale transform $y_t = \mu_{x_t} + R z_t$ of an

innovation z_t . The density function of z_t is given by $h_K(z|x) \propto [H(z, x)]^2 \phi(z)$ where

$$H(z, x) = \sum_{i=0}^K \theta_i(x) z^i = \sum_{i=0}^{K_z} \left(\sum_{j=0}^{K_x} a_{ij} x^j \right) z^i \text{ and } \phi(z) \text{ denotes the multivariate standard}$$

normal density function. K_x and K_z are the polynomial orders of x and z respectively.

Normalizing the first coefficient $a_{00} = 1$ we can approximate any density function as close as possible by a large enough truncated series with leading term a standard normal density.

With this set up, SNP is a nonparametric model that incorporates the familiar Gaussian VAR(q) model (setting the conditional mean equation to be VAR(q) process, the conditional variance R a constant and $K_z = K_x = 0$), semiparametric VAR(p) model (allowing a positive K_z), the Gaussian ARCH(p,q) model (conditional mean

following an AR(p) process with $K_x > 0$ or letting the conditional variance R be a function of x , $K_z = 0$), the semiparametric ARCH model (allowing a positive K_z), the Gaussian GARCH model (adding a Gaussian GARCH leading term to the conditional variance R , and $K_z = K_x = 0$), the semiparametric GARCH model (allowing a positive K_z), and a nonparametric model (allowing positive K_z and K_x).

When we have more than two state variables the interaction terms in the function of $H(z, x)$ become too burdensome. SNP allows users to have two additional tuning parameters, I_x and I_z , to control the number of interactions. A positive I_z means all interactions of order larger than $K_z - I_z$ are excluded from the equation; similarly for a positive I_x . To summarize, the tuning parameter set includes:

$L_\mu, (L_r, L_g), (L_H, K_z, I_z, K_x, I_x)$, where L_μ is the AR order of the conditional mean,

L_r is the ARCH-like order for the conditional variance (see Gallant & Tauchen (2001) for references), L_g is the GARCH-like order for the conditional variance, L_H is the

number of lags that go into the x part of the polynomial function of $H(z, x)$,

K_z, K_x are orders of z and x respectively in $H(z, x)$, I_x and I_z are as explained.

Different combinations of tuning parameters correspond to different types of models.

Refer to Chernov & Ghysels (2000) Table 1 panel B for taxonomy of SNP models.

The S&P 500 index data is used in this dissertation to estimate the Heston model parameters. The data range is from March 14, 1986 to February 29, 2008. The reason we choose this period is that it includes major historical financial crisis such as 1987 market crash, 1998 Asian financial crisis, 2000 internet bubble and the current

subprime mortgage crisis. Statistical descriptions of the data are listed in Table 3 and graphed in Figure 15:

Table 3 - Statistics for S&P 500 Index Daily Log-returns from 03/14/1986 to 02/29/2008

Sample Quantiles:				
min	1Q	median	3Q	max
-0.229	-0.004579	0.0005742	0.00561	0.08709
Sample Moments:	mean	std	skewness	kurtosis
	0.0003118	0.01072	-1.98	45.02
Number of Observ.	5539			

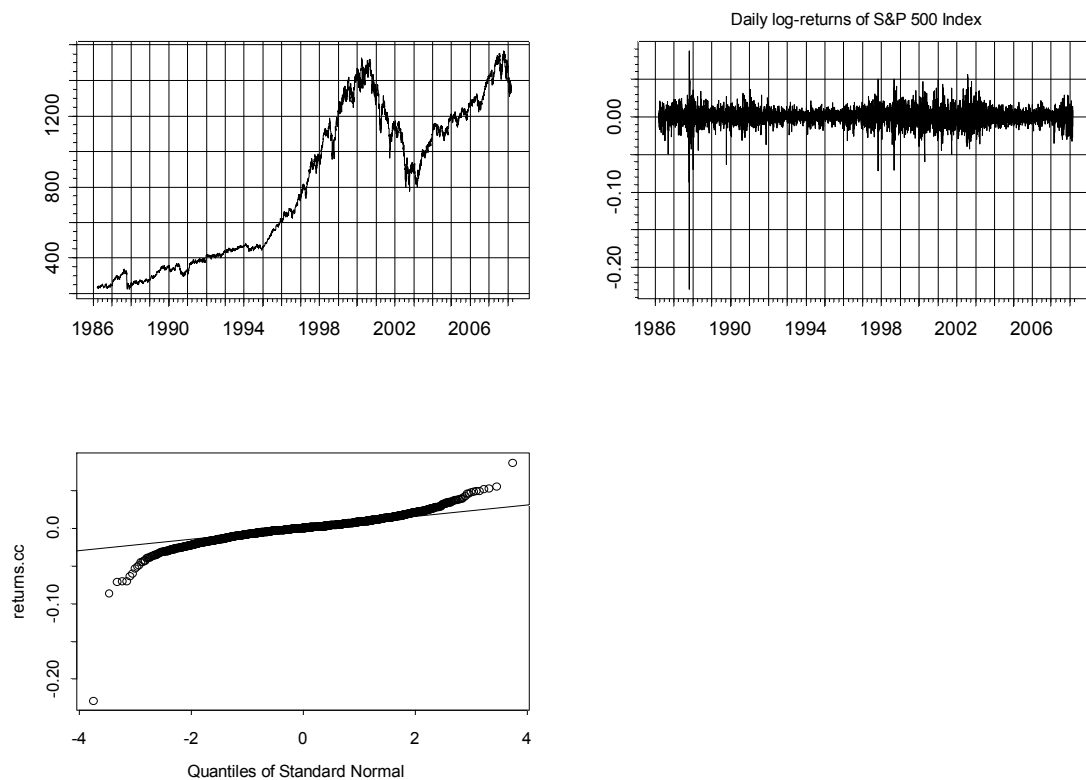


Figure 15 - Summary Statistics Graph for S&P 500 Index during the Period

From left to right, up to down are time series of daily S&P 500 index, the log-return series of S&P 500 index and the Q-Q plot of the daily return against standard normal quantiles.

4.1.2-Fitted Density of Data

In the method of SNP, optimal tuning parameters are selected according to values of the Akaike information criterion (AIC), The Hannan and Quinn criterion (HQ) and the Schwartz Bayes information criterion (BIC). We will only report BIC here as various articles (e.g. Fenton & Gallant (1996), (1997), Gallant & Tauchen (2001) have reached a consensus that BIC performs best in determining the best fitting model . Gallant & Tauchen (2001) also suggests a specification search procedure: starting from the initial model $(L_\mu, L_g, L_r, L_H, K_z, I_z, K_x, I_x) = (1, 1, 1, 1, 0, 0, 0, 0)$ then K_z expanded in increments of 2, up to a maximum of 8, until the BIC indicates no further expansion needed, next increasing L_g, L_r to allow ARCH/GARCH-like effect on conditional variance, lastly, increasing K_x to unity to allow state dependence of polynomial coefficients.

Based on the above suggestion (we implement the search procedure as follows:

$(1, 1, 1, 1, 0, 0, 0, 0) \rightarrow (1, 1, 1, 1, 2, 0, 0, 0) \rightarrow (1, 1, 1, 1, 4, 0, 0, 0) \rightarrow (1, 1, 1, 1, 6, 0, 0, 0) \rightarrow (1, 1, 2, 1, 6, 0, 0, 0)$

$\rightarrow (1, 1, 1, 1, 8, 0, 0, 0) \rightarrow (1, 1, 1, 1, 8, 0, 1, 0) \rightarrow (1, 1, 1, 1, 8, 0, 2, 0)$

In addition to the above, we also try the auto search function `SNP.auto(.)` provided by

S-PLUS, the tuning parameter set in Chernov & Ghysels (2000) and Gallant &

Tauchen (2001). The parameters and associated BICs are reported in Table 4.

Table 4 - Comparison of Different SNP Models

SNP Model	L_μ	L_g	L_r	L_H	K_z	I_z	K_x	I_x	BIC
10010000	1	0	0	1	0	0	0	0	1.4214
11110000	1	1	1	1	0	0	0	0	1.2769
11114000	1	1	1	1	4	0	0	0	1.2453
11116000	1	1	1	1	6	0	0	0	1.2413
01118000*	0	1	1	1	8	0	0	0	1.2383
02118000	0	2	1	1	8	0	0	0	1.2389
11118000	1	1	1	1	8	0	0	0	1.2390
11118010	1	1	1	1	8	0	1	0	1.2416
11118020	1	1	1	1	8	0	2	0	1.2452
20b14000**	2	0	11	1	4	0	0	0	1.2552
31819000***	3	1	8	1	9	0	0	0	1.2438

*: the best fitting by S-PLUS auto search function; **: the best fitting by Gallant & Tauchen (2001);

***: the best fitting by Chernov & Ghysels (2000)

Judging by the BIC the best fitting model is (0,1,1,1,8,0,0,0) selected through the auto search function in S-PLUS. Before we use it for EMM, we should still conduct some model diagnostic checks to ensure that the chosen model is adequate. Figure 16 to Figure 20 present the model diagnostic results.

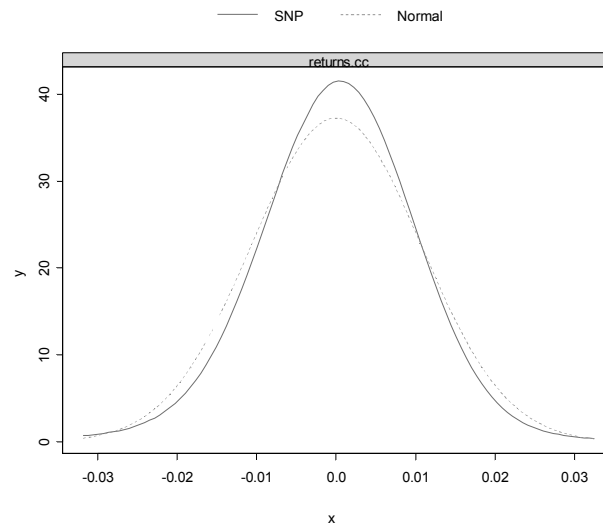


Figure 16 - Fitted SNP Density of Daily Log-returns of S&P 500 Index

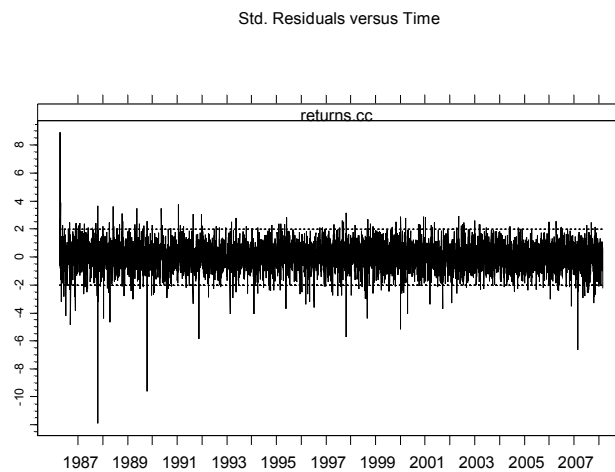


Figure 17 - Standardized Residuals from SNP Fitting to Daily Log-returns of S&P 500 Index

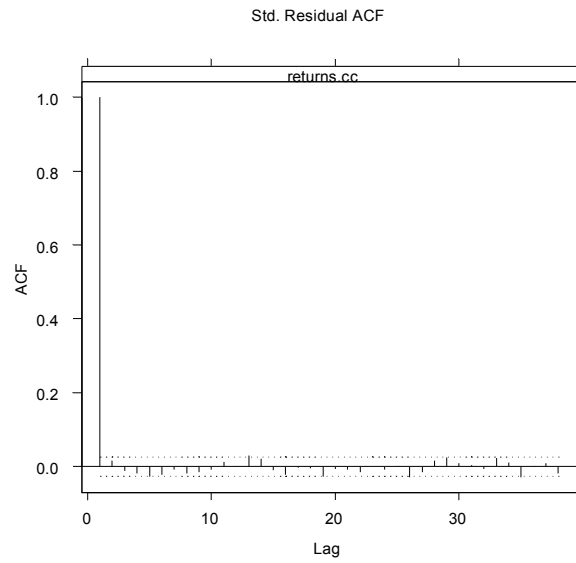


Figure 18 - ACF for Standardized Residuals from SNP Fitting to Daily Log-returns of S&P 500 Index

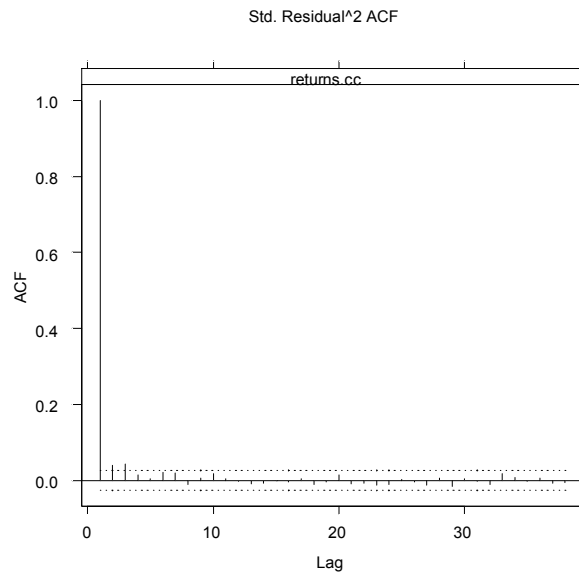


Figure 19 - ACF for Squared Standardized Residuals from SNP Fitting to Daily Log-returns of S&P 500 Index

The figure shows that the SNP density has captured the autocorrelation in the conditional heteroskedasticity present in the return data.

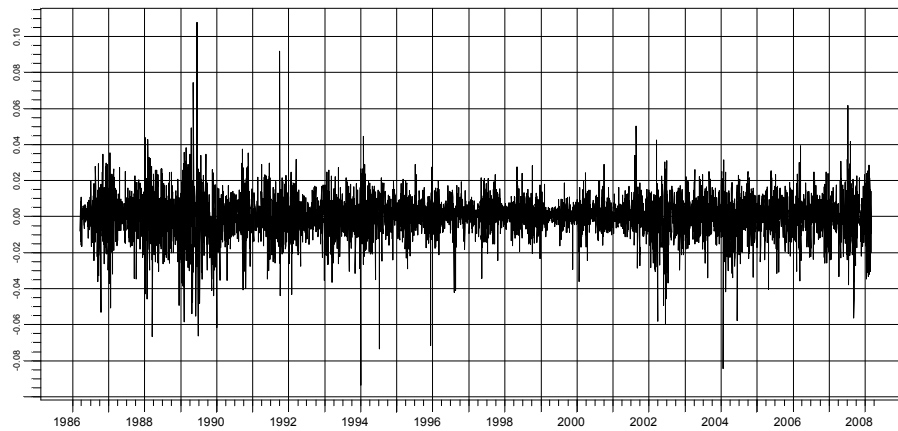


Figure 20 - Simulated Returns from SNP Density Fitted to Daily Log-returns of S&P 500 Index

The above model checking shows that the fitted density of $(0, 1, 1, 1, 8, 0, 0, 0)$ is adequate for the data, hence can be safely used in EMM in the next section.

Section II - EMM and Reprojection Method

Modern finance and economics models often involve state variables that are not directly observable. Examples include continuous time SVM, interest rate models and general equilibrium models. Standard statistical methods such as Maximum

Likelihood Estimates (MLE) and Bayesian are not suitable in these circumstances since there are generally no explicit likelihood functions available. EMM is created exactly to tackle these types of problems.

Implementation of EMM involves two steps: the first step we just completed- summarize the data by projecting the observed data onto a transition density which is the SNP density. Once we get the SNP density, its score will be used as the score generator for EMM; the second step requires simulation to evaluate the expectation of the score for a given set of parameters and compute a chi-squared criterion function. Then a nonlinear optimizer is used to find parameters that minimize this criterion. Theory (refer to Gallant & Long (1997)) guarantees that as long as the SNP density can capture all characteristics of the data generating process, then EMM is fully efficient.

4.2.1-EMM

Let us presume that we already have a SNP density at hand which is in the form of:

$$(0.38) \quad f_K(y|x, \rho) = \frac{h_K[R_x^{-1}(y - \mu_x)|x]}{\det(R_x)}$$

Recall we did a location-scale transform $y_t = \mu_{x_t} + Rz_t$ of an innovation z_t which has density function given by $h_K(z|x)$. Out of (7.1) a score generator function can be derived as:

$$(0.39) \quad \phi_f^0(y) = \frac{\partial}{\partial \theta} \log f(y|\rho^0)$$

where $\rho^0 = \arg \max_{\rho} : E[\log f(\cdot|\rho)]$

And the weighting matrix for GMM estimator is:

$$(0.40) \quad \tilde{S} = \frac{1}{n} \sum_{i=1}^n \phi_f^0(x, y) \phi_f^{0-1}(x, y)$$

With (8.1) and (8.2) our first step of projecting data is completed. Now let θ denote the parameter set we try to estimate. The moment function

$$(0.41) \quad m_n(\theta) = E_{\theta}[\varphi_n], \text{ n is the sample size}$$

The parameters then can be estimated as follows:

$$(0.42) \quad \tilde{\theta}_n = \arg \min_{\theta} : m_n(\theta)' \tilde{S}^{-1} m_n(\theta)$$

Asymptotically the estimator given by (8.4) is consistent and normal and under regularity conditions it is efficient. Best of all, hypothesis testing and statistical model diagnostics can be performed (see Gallant & Tauchen, 1996; Gallant & Long, 1997, for further references).

4.2.2-Reprojection Method

Another big advantage of using EMM for our purpose is that it provides another way of doing volatility filtering. We are interested in the hidden volatility because initial

volatility level is required in the Heston pricing equation to extract the market prices of risks. We will use the recovered hidden volatility through the reprojection method as the initial volatility.

The idea of reprojection is as follows (Gallant & Tauchen (2001)). During the process of EMM we have generated series of both hidden state variables $\{\hat{v}_t\}$, observables $\{\hat{y}_t\}$ through simulation. We can fit a SNP-GARCH model on the $\{\hat{y}_t\}$ to get the one-step ahead conditional variance $\{\hat{\sigma}_t^2\}$ of $\{\hat{y}_{t+1}\}$ given $\{\hat{y}_t\}$. Based on these series a functional form of the conditional distribution of hidden variables $\{\hat{v}_t\}$ given observables $\{\hat{y}_t\}$, $\{\|\hat{y}_t\|\}$, $\{\hat{\sigma}_t^2\}$ and their lags can be obtained by regression. Lastly we simply plug in the real observed data to get the hidden processes.

As aforementioned, we will use EMM to estimate the model parameters such as the drift rate, the mean reversion rate etc. It is worthy to point out that the EMM estimates, along with estimates using other statistical methods, are subject to the time period one is looking at. That is, if we use a different set of data, we may have different EMM parameter estimates. This can potentially affect the results we get for risk premium and risk aversion. However, we argue that this is of little concern for us in this dissertation because we use a relatively long dataset with more than 20 years of daily prices and our preliminary tests have shown no big differences between the estimates.

Section III - Quadratic Exponential Martingale (QEM) Corrected Scheme for Heston Model

As the EMM and SNP both are simulation based methods and the model we are concerned with is a continuous time diffusion model such as Heston SVM, we have to examine the discretization scheme used in SNP and EMM. Previous studies usually adopt either Euler scheme or weak order 2 convergence scheme. However, recent studies have found that most current schemes including the naive Euler scheme and weak order schemes such as implicit Milstein only work under a small range of parameter combinations while performing poorly for real data. Using truncations only cause larger bias. See Andersen (2006) and Zhu (2008) for detailed discussions.

Andersen (2006) provides a new discretization method QEM that seems to work pretty well under most of the real world parameter combinations. In this section we will only provide the main idea involved in the QEM algorithm. For details and theoretical results readers are referred to his article.

Observing the kink in the cumulative distribution of the volatility process (Figure 1 in his article), Andersen proposes to adopt a quadratic function of standard normal variable as proxy to the non-central chi-squared distribution of the volatility for moderate and high values of volatility, and use an 0-centered probability mass combined with an exponential tail for low values of volatility to proxy the central chi-squared distribution around zero. For the underlying asset process, Andersen points out the “leaking coefficient” problem in Euler scheme which may lead to serious mispricing for at-the-money options. He proposes to start from the exact

representation of the (log) underlying price and use natural discretization for the involved terms. In the final step he makes martingale corrections to the scheme such that the log price process remains a martingale. Figure 4.7 shows simulated return series for different discretization schemes. It is easy to visualize that the simulated series using a combination of asset and square root of volatility or logarithm of asset and logarithm of volatility both fail in this case. The QEM simulated series, however, do capture some volatility clustering and excess kurtosis.

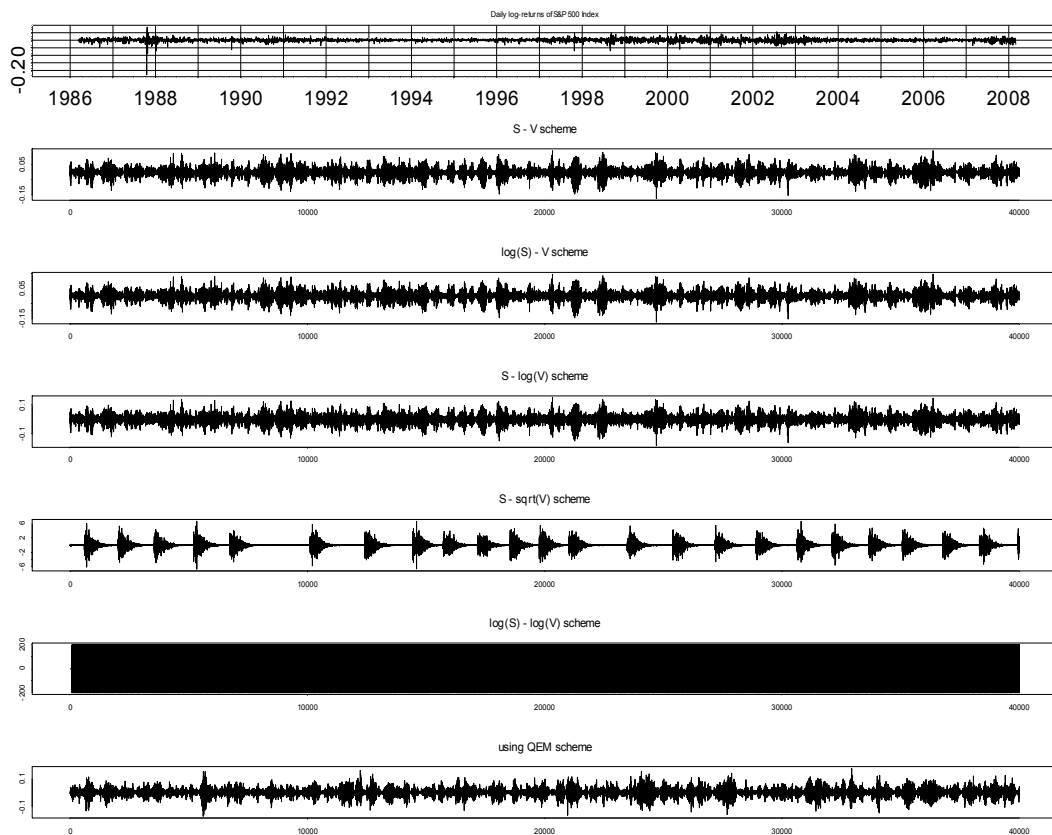


Figure 21 - Simulated Return Series with Different Discretization Schemes

Figure 21: Different simulated return series with different discretization schemes.

Asset-Volatility, Log(asset)-Volatility, Asset-log(volatility), Asset-sqrt(volatility), log(asset)-log(volatility) and QEM. The parameters used for simulations are the same: $\rho = (0.227, 2.2254, 0.12013, 0.9012, -0.564)$ corresponding to the drift rate, mean reversion rate, mean reversion level, volatility of volatility and correlation coefficient respectively.

We implement the QEM in S-PLUS and find good performances. To combine this recently devised QEM with EMM is a fairly new attempt and we expect the combination should give more stable and reliable estimates.

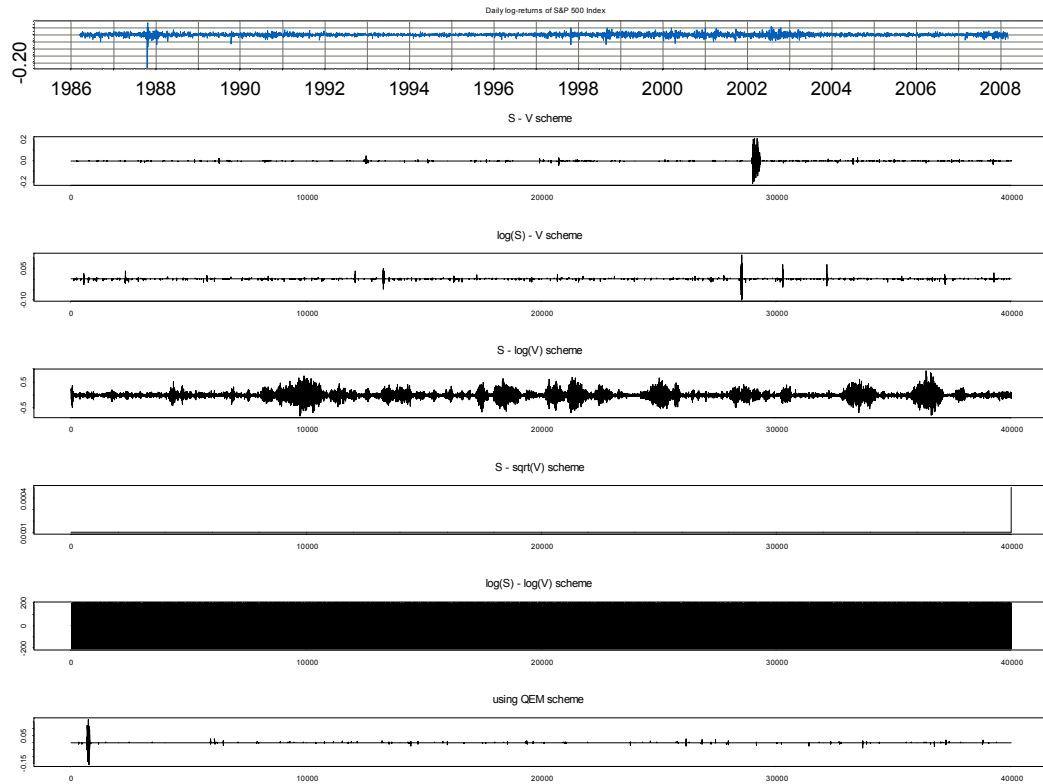


Figure 22 - Another Simulated Return Series with Different Discretization Schemes

Figure 22: Different simulated return series with different discretization schemes.

Asset-Volatility, Log(asset)-Volatility, Asset-log(volatility), Asset-sqrt(volatility), log(asset)-log(volatility) and QEM. The parameters used for simulations are the same: $\rho = (0.527, 0.2254, 0.32013, 2.9012, -0.264)$ corresponding to the drift rate, mean reversion rate, mean reversion level, volatility of volatility and correlation coefficient respectively.

It should be warned, however, that while the QEM works well in most of the cases it certainly does not work all the time. For example, Figure 4.8 shows such a case where all but one algorithm actually gives somewhat reasonable process.

Section IV - EMM estimates under Heston SVM for the S&P 500 index

Recall that in this section we consider the Heston SVM as specified below:

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{Y_t} S_t dW_t \\ dY_t &= \kappa(\theta - Y_t)dt + \xi \sqrt{Y_t} d\tilde{Z}_t \\ dW_t d\tilde{Z}_t &= \rho_t dt \end{aligned}$$

The parameter set is $\psi = (\mu, \kappa, \theta, \xi, \rho)$. We implement EMM using the emm function in S-PLUS and allow random perturbations to ensure we get a global minimum. The results are reported in Table 5.

Table 5 - EMM Estimates for Heston SVM

initial parameter guesses in EMM algorithm

rho.emm=c(0.0431,2.07356,0.0284,0.2273,-0.31)

Coefficients:

	Value	Std. Error	95% Conf. Int.	
mu	0.09235	0.040850	-0.005883	0.16963
kappa	2.51057	0.789307	2.490266	3.07687
theta	0.02826	0.003669	0.021857	0.03592
xi	0.27964	0.043171	0.232572	0.32739
corr	-0.64175	0.122236	-0.705703	-0.54257

Final optimization:

Convergence: relative function convergence

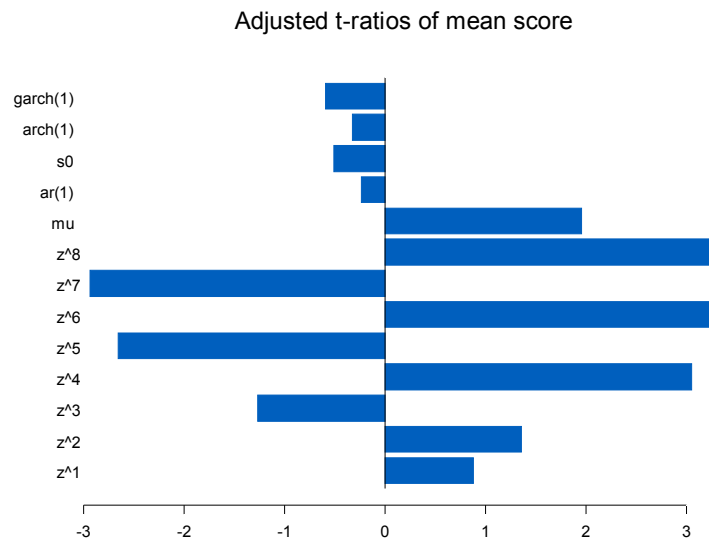
Iterations: 43

EMM objective at final iteration: 26.26

P-Value: 0.0009486 on 8 degrees of freedom

Note: all reported estimates are annual after transformation although the original data are daily. The objective function has a large value 26.26 and a relatively small p-value, indicating that the model is not significant and is rejected by the data. Table 6 identifies those factors that may cause the failure of our model.

Table 6 - Adjusted Ratios of Mean Score for EMM



Section V – Conclusion

In this chapter we introduced SNP and EMM methods for parameter estimation in the Heston model. To get better empirical results we borrowed Andersen's QEM algorithm to discretize the continuous time model. The SNP density obtained seems to be able to capture all important characteristics of the data and the EMM indicates that the model is inadequate. This is consistent with several previous studies conclusion that a simple one volatility factor model such as Heston model is not good enough to capture all the dynamical time series of the underlying data. We will use the estimates presented in Table 4.3 in the next section to extract market prices of risks and the market risk aversion.

APPENDIX 4.1: S-PLUS CODE FOR EMM AND QEM

```
# script for getting the return series of S&P 500 index #
# data are stored in table.csv
# data range: March 14, 1986 to February 29, 2008      #
#####

# step0: change the "factor" feature of table.dat to    #
# "character", then transform the dataset to time series#

td=timeDate(table[,1],in.format="%Y-%m-%d",format="%b %d, %Y")
sp500.ts=timeSeries(pos=td,data=table.df[,-1])
sp500.index.ts=timeSeries(pos=td,data=table.df[,"Adj.Close"])
sp500.index.ts=rev(sp500.index.ts)

# Now get the return series out the index series      #

returns.cc=getReturns(sp500.index.ts,type="continuous")
returns.cc@title="Daily log-returns of S&P 500 Index"

# Describe statistical attributes of the data          #

summaryStats(returns.cc)
par(mfrow=c(2,2))
plot(sp500.index.ts)
plot(returns.cc)
qqnorm(returns.cc)
qqline(returns.cc)
histPlot(returns.cc)

# Derive ACF, PACF of the return series              #
```

```

par(mfrow=c(2,2))
acf.return=acf(returns.cc)
acf.return=acf(returns.cc^2)
pacf.return=acf(returns.cc,type="partial")
pacf.return=acf(returns.cc^2,type="partial")

# Now, start fitting the SNP density to #
# the return series #

fOld=c(0,1e-5,0,-1e-5,1e-5,1e-5,1e-4,0,-1e-4,
+ 1e-4,1e-4,1e-3,0,-1e-3,1e-3,1e-3,1e-5,0,
+ -1e-2,1e-2,1e-2,1e-1,0,-1e-1,1e-1,1e-1,
+ 1e0,0,-1e0,1e0,1e0)

fNew=c(0,0,1e-5,1e-5,-1e-5,1e-5,0,1e-4,1e-4,
+ -1e-4,1e-4,0,1e-3,1e-3,-1e-3,1e-3,0,
+ 1e-2,1e-2,-1e-2,1e-2,0,1e-1,1e-1,-1e-1,
+ 1e-1,0,1e0,1e0,-1e0,1e0)

fit.returns.10010000=SNP(returns.cc,model=SNP.model(ar=1),
+ control=SNP.control(xTransform="logistic",
+ fOld=fOld,fNew=fNew),
+ n.drop=14)

# This is the auto fit by S-plus #

fit.returns.auto=SNP.auto(returns.cc,
control=SNP.control(xTransform="logistic",
seed=011667,n.start=n.start,fOld=fOld,fNew=fNew),
n.drop=14)

# Alternatively, we consider expansion #
# from 10010000 to 11110000 to 11114000 #
# to 11116000 to 11118000 to 11118010 #

```



```

# to 11118020 to 11216000                                #

fit.returns.11110000=expand(fit.returns.10010000,arch=1,garch=1,trace=T)
fit.returns.11114000=expand(fit.returns.11110000,zPoly=4)
fit.returns.11116000=expand(fit.returns.11114000,zPoly=2)
fit.returns.11118000=expand(fit.returns.11116000,zPoly=2)
fit.returns.11118010=expand(fit.returns.11118000,xPoly=1)
fit.returns.11118020=expand(fit.returns.11118010,xPoly=1)
fit.returns.11216000=expand(fit.returns.11116000,arch=1)

# Turns out that the auto selection gives                #
# the best fitting model: 01118000                        #

# model Diagnostics                                       #
#####

# simulation test                                          #

sim.returns.auto=simulate(fit.returns.auto)
plot(sim.returns.auto)

SNP.density(fit.returns.auto)

# Residual analysis                                       #

plot(fit.returns.auto)                                # select 1 to plot all

# Now we test EMM in BSM case using real data           #

rho.BSM.emm=c(0.0341,0.892)                            # initial parameter guesses in EMM
algorithm
rho.BSM.emm.names=c("mu","sigma")
n.BSM.sim=100000

```

```

n.BSM.burn=1000
ndt.BSM=25
n.start=25
N.BSM=1
M.BSM=1
t.BSM.per.sim=1/252
set.seed(567)
z.BSM=rnorm(M.BSM*ndt.BSM*(n.BSM.sim+n.BSM.burn))
X0.BSM=278

tf.BSM.emm.aux=euler.pcode.aux(rho.names=rho.BSM.emm.names,
                                drift.expr=expression(mu*X),
                                diffuse.expr=expression(sigma*X),

                                N=N.BSM,M=M.BSM,t.per.sim=t.BSM.per.sim,ndt=ndt.BSM,returnc=T,
                                lbound=0,ubound=3000,z=z.BSM,X0=X0.BSM)

emm.BSM.fit.test=EMM(fit.returns.auto,coef=rho.BSM.emm,
                      control=EMM.control(n.burn=n.BSM.burn,n.sim=n.BSM.sim,
                                           initial.itmax=10,final.itmax=300,n.start=rep(5,5),
                                           tweak=c(0.1,.25,.50,0.75,1)),
                      gensim.fn="BSM.gensim",
                      gensim.language="SPLUS",
                      gensim.aux=tf.BSM.emm.aux)

names(emm.BSM.fit.test$coef)=rho.BSM.emm.names
emm.BSM.fit.test
plot(emm.BSM.fit.test)

```

```
#####

# Simulator Function          #
# continuous stochastic volatility  #
#  $dS = \mu \cdot Sdt + \sqrt{V} \cdot S \cdot dW_t$           #
#  $dV = \kappa \cdot (\theta - V)dt + \xi \cdot \sqrt{V} \cdot dZ$  #
#####

rho=c(0.227, 2.2254, 0.12013, 0.9012, -0.564) # parameters for simulation
rho.names=c("mu","kappa","theta","xi","corr")

N=2          # number of state variables
M=2          # number of driving Brownian
              motions

ndt=25       # time segments for each time
              interval

n.sim=40000  # simulation times
n.burn=1000  # number of burn-ins

set.seed(3253)

z=rnorm(M*ndt*(n.sim+n.burn)) # number of needed random
                                variables

t.per.sim=1/252

X0.std=c(278,0.02)          # initial value of state variables
                             for standard discretization

X0.sqrtvol=c(5.628,0.1414) # initial value of state
                             variables for milstein
                             discretization

X0.logvol=c(278,6-3.912)    # initial value of state
                             variables for log vol
                             discretization

X0.doublelog=c(5.628,-3.912) # initial value of state
                              variables for double log
                              discretization
```

```

X0.logS=c(5.628,0.02)                                # initial value of state
                                                         variables for logS
                                                         discretization
X0.QEM=c(268,0.02)                                    # initial value of state
                                                         variables for QEM
                                                         discretization

#####
# Simulator Function                                #
# using various Schemes                            #
#####

# I. this one uses sqrt of vol process #

tf.sqrtvol.aux=weak2.pcode.aux(rho.names=rho.names,
                                drift.expr=expression(mu-
0.5*max(0,X[2])^2,(kappa*(theta-max(0,X[2])^2)-
0.25*xi^2)/(2*max(0,X[2]))),
                                diffuse.expr=expression(max(0,X[2]),0,
0.5*xi*corr,0.5*sqrt(1-corr^2)*xi),
                                N=N,M=M,t.per.sim=t.per.sim,ndt=ndt,returnc=c(T,F),
                                z=z,X0=X0.sqrtvol)

#####
#####
# the following is just a test of simulated asset price and vol process #
# remember to change back to "return=c(T,F)" after the test                #

#tf.sim=euler.pcode.gensim(rho=rho,n.sim=n.sim,n.var=2,n.burn=n.burn,aux=tf.m
ilstein.aux)
#tmp=matrix(tf.sim,n.sim,2,byrow=T)
#tmp=tmp[,1]
#tsplot(tmp)
#tmp=tmp[,2]

```

```

#tsplot(tmp)

#####

Heston.sqrtvol.gensim=function(rho,n.sim,n.var,n.burn,aux)
{
  tf.sim=weak2.pcode.gensim(rho=rho,n.sim=n.sim,n.var=n.var,n.burn=n.burn,a
ux=aux)
  tmp=matrix(tf.sim,n.sim,n.var,byrow=T)
  tmp=diff(tmp[,1],lag=1)
  tmp=append(tmp,0.0005,after=length(tmp)) # to restore the length to
n.sim
}

Heston.sqrtvol.sim=Heston.sqrtvol.gensim(rho,n.sim,1,n.burn,tf.sqrtvol.aux)
# this returns the simulated return series

summaryStats(Heston.sqrtvol.sim)
tsplot(Heston.sqrtvol.sim)
qqnorm(Heston.sqrtvol.sim)
qqline(Heston.sqrtvol.sim)

# Experiments show that the parameter "corr" controls "skewness"
# While parameter "xi" controls "kurtosis"
# Hence to fit the data well we need a very negative "corr"
# and a large value of "xi". In fact, people propose to model "xi"
# Moreover, looks like a negative "corr" combined with a large "xi"
# produces bigger effect than either single factor
# An interesting observation is that "mu" affects both "skewness"
# and "kurtosis". Higher "mu" produces more negative skewness
# and higher kurtosis.

# II. this one uses log volatility SDE #
# # # # # # # # # # # # # # # # # # # # # #

```

```

tf.logvol.aux=weak2.pcode.aux(rho.names=rho.names,
                             drift.expr=expression(mu*X[1], (kappa*(theta-exp(X[2]-
6))-0.5*xi^2)/exp(X[2]-6)),
                             diffuse.expr=expression(sqrt(exp(X[2]-6))*X[1], 0,
                             xi*corr/sqrt(exp(X[2]-
6)), sqrt(1-corr^2)*xi/sqrt(exp(X[2]-6))),

N=N,M=M,t.per.sim=t.per.sim,ndt=ndt,returnc=c(T,F),lbound=1,ubound=3000,
z=z,X0=X0.logvol)

Heston.logvol.gensim=function(rho,n.sim,n.var,n.burn,aux)
{
  tf.sim=weak2.pcode.gensim(rho=rho,n.sim=n.sim,n.var=n.var,n.burn=n.burn,aux=aux)
  tmp=matrix(tf.sim,n.sim,n.var,byrow=T)
  tmp=getReturns(tmp[,1],type="continuous")
  tmp=append(tmp,0.0005,after=length(tmp)) # to restore the length to
  n.sim
}

Heston.logvol.sim=Heston.logvol.gensim(rho,n.sim,1,n.burn,tf.logvol.aux)
# this returns the simulated return series

summaryStats(Heston.logvol.sim)
tsplot(Heston.logvol.sim)
qqnorm(Heston.logvol.sim)
qqline(Heston.logvol.sim)

# III. this one uses standard SDE #
# # # # # # # # # # # # # # # # # # #

tf.standard.aux=weak2.pcode.aux(rho.names=rho.names,
                             drift.expr=expression(mu*max(0,X[1]),kappa*(theta-
max(0,X[2]))),
                             diffuse.expr=expression(sqrt(max(0,X[2]))*X[1],0,

```

```

sqrt(max(0,X[2]))*xi*corr,sqrt(max(0,X[2]))*sqrt(1-corr^2)*xi),
      N=N,M=M,t.per.sim=t.per.sim,ndt=ndt,returnc=c(T,F),
      z=z,X0=X0.std)

Heston.std.gensim=function(rho,n.sim,n.var,n.burn,aux)
{
  tf.sim=weak2.pcode.gensim(rho=rho,n.sim=n.sim,n.var,n.burn=n.burn,aux=aux)
  tmp=matrix(tf.sim,n.sim,n.var,byrow=T)
  tmp=getReturns(tmp[,1],type="continuous")
  tmp=append(tmp,0.0005,after=length(tmp)) # to restore the length to
  n.sim
}

Heston.std.sim=Heston.std.gensim(rho,n.sim,1,n.burn,tf.standard.aux) #
  this returns the simulated return series

summaryStats(Heston.std.sim)
tsplot(Heston.std.sim)
qqnorm(Heston.std.sim)
qqline(Heston.std.sim)

# IV. this one uses double log SDE #
# # # # # # # # # # # # # # # # # #

tf.doublelog.aux=weak2.pcode.aux(rho.names=rho.names,
      drift.expr=expression(mu-0.5*exp(X[2]),(kappa*(theta-
exp(X[2]))-0.5*xi^2)/exp(X[2])),
      diffuse.expr=expression(sqrt(exp(X[2])),0,

      xi*corr/sqrt(exp(X[2])),sqrt(1-corr^2)*xi/sqrt(exp(X[2]))),
      N=N,M=M,t.per.sim=t.per.sim,ndt=ndt,returnc=c(T,F),
      z=z,X0=X0.doublelog)

Heston.doublelog.gensim=function(rho,n.sim,n.var,n.burn,aux)

```



```

k4 = gamma2 * dtyr * (1 - rho[5] * rho[5])
capA = k2 + 0.5 * k4
k13 = k1 + 0.5 * k3

var0 = 0.02
S0=268
Vt = var0
lnSt =log(S0)

n=n.sim+n.burn

SimArr=matrix(0,n.sim,2)

# start generating the simulations

for (i in 1 : n)
{

M = rho[3] + (Vt - rho[3]) * ekd
V = Vt * v1 + v2
psi = V / (M * M)
uv = aux$zV[i]

if (psi <= psiC) {
  psi2 = 2 / psi
  b2 = max(psi2 - 1 + sqrt(psi2) * sqrt(psi2 - 1), 0)
  a = M / (1 + b2)
  qV = qnorm(uv)
  b = sqrt(b2)

  capAa = capA * a

  if (is.na(capAa) || capAa>=0.5) { } # do nothing if capAa is not a number
  or greater than 0.5
else {

```

```

      k0 = -capAa * b2 / (1 - 2 * capAa) + 0.5 * log(1 - 2 * capAa) - k13 *
Vt    # refer to eqn.(41)in Leif Andersen
      VtdT = a * (b + qV) * (b + qV)

      qS = qnorm(aux$zS[i])

      lnStdT = lnSt + rho[1] * dtyr + k0 + k1 * Vt + k2 * VtdT + qS * sqrt(k3
* Vt + k4 * VtdT)

      lnSt = lnStdT

      Vt = VtdT

      # store the result in each simulation step to an array/matrix
      if (i>n.burn) {
        SimArr[i-n.burn,1]=lnSt
        SimArr[i-n.burn,2]=Vt
      }

    }
  }

else

{
  p = (psi - 1) / (psi + 1)
  beta = (1 - p) / M

  if (is.na(capA) || capA>=beta){} # do nothing if capA is not a number or
greater than beta
  else {
    k0 = -log(p + beta * (1 - p) / (beta - capA)) - k13 * Vt    # refer to
eqn.(43)in Leif Andersen

```

```

    if (uv <= p)
      {VtdT = 0}
    else
      {VtdT = log((1 - p) / (1 - uv)) / beta}

    qS = qnorm(aux$zS[i])

    lnStdT = lnSt + rho[1] * dtyr + k0 + k1 * Vt + k2 * VtdT + qS * sqrt(k3
* Vt + k4 * VtdT)

    lnSt = lnStdT

    Vt = VtdT

    # store the result in each simulation step to an array/matrix
    if (i>n.burn) {
      SimArr[i-n.burn,1]=lnSt
      SimArr[i-n.burn,2]=Vt
    }
  }
}

} # end of for loop

tmp=diff(SimArr[,1],lag=1)
tmp=append(tmp,0.0005,after=length(tmp)) # to restore the length to
n.sim
}

Heston.QEM.aux=list(zV=runif(n.sim+n.burn,min=0,max=1),zS=runif(n.sim+n.burn,
min=0,max=1))

```

```

Heston.QEM.sim=Heston.QEM.gensim(rho,n.sim,1,n.burn,Heston.QEM.aux)      #
    this returns the simulated return series

summaryStats(Heston.QEM.sim)
tsplot(Heston.QEM.sim)
qqnorm(Heston.QEM.sim)
qqline(Heston.QEM.sim)

# compare the simulated data across the above #
# SIX schemes.                                #

par(mfrow=c(7,1))
plot(returns.cc)
tsplot(Heston.std.sim,main="S - V scheme")
tsplot(Heston.logS.sim,main="log(S) - V scheme")
tsplot(Heston.logvol.sim,main="S - log(V) scheme")
tsplot(Heston.sqrtvol.sim,main="S - sqrt(V) scheme")
tsplot(Heston.doublelog.sim,main="log(S) - log(V) scheme")
tsplot(Heston.QEM.sim,main="using QEM scheme")
par(mfrow=c(1,1))

#####
# if one increases the value xi, then one gets      #
# more realistic (i.e.higher kurtosis) simulated data.#
# this inspires including the third factor to SVM    #
#####

# we test SNP in simulated data first                #

fit.Heston.sim=SNP.auto(Heston.QEM.sim,control=SNP.control(xTransform="logist
    ic",
    seed=011667,n.start=n.start,fOld=fOld,fNew=fNew), n.drop=14)

fit.Heston.sim

```

```

# snp diagnostics
SNP.density(fit.Heston.sim)

snp.Heston.resid = residuals(fit.Heston.sim,standardized=T)
par(mfrow=c(3,1))
tsplot(snp.Heston.resid,main="standardized residuals")
abline(h=c(2,-2))
tmp = acf(snp.Heston.resid)
tmp = acf(snp.Heston.resid^2)
par(mfrow=c(1,1))

# simulate from fitted SNP model
set.seed(123)
snp.Heston.sim = simulate(fit.Heston.sim)
tsplot(snp.Heston.sim,main="simulation from SNP 01118000 model")

#####
#####

rho.emm=c(0.14884,2.33262,0.02538,0.54579,-0.43027) # initial
parameter guesses in EMM algorithm
rho.emm.names=c("mu","kappa","theta","xi","corr")
n.sim=60000
n.burn=1000
set.seed(456)
Heston.QEM.aux=list(zV=runif(n.sim+n.burn,min=0,max=1),zS=runif(n.sim+n.burn,
min=0,max=1))

emm.Heston.QEM.fit=EMM(fit.Heston.sim,coef=rho.emm,
control=EMM.control(n.burn=n.burn,n.sim=n.sim,
initial.itmax=10,final.itmax=300,n.start=rep(5,3),
tweak=c(.25,.50,1)),
gensim.fn="Heston.QEM.gensim",
gensim.language="SPLUS",

```

```

gensim.aux=Heston.QEM.aux)

names(emm.Heston.QEM.fit$coef)=rho.emm.names
emm.Heston.QEM.fit
plot(emm.Heston.QEM.fit)

#####

# Now we fit the real return series data #
#####

rho.emm=c(0.0431,2.7356,0.284,0.273,0) # initial parameter
      guesses in EMM algorithm
rho.sp500.emm=c(0.09235,2.51057,0.02826,0.27964,-0.64175)
rho.sp500.emm.names=c("mu","kappa","theta","xi","corr")
n.sp500.sim=60000
n.sp500.burn=100
set.seed(257)
Heston.sp500.QEM.aux=list(zV=runif(n.sp500.sim+n.sp500.burn,min=0,max=1),zS=r
      unif(n.sp500.sim+n.sp500.burn,min=0,max=1))

## use the 11118000 fit function ##

emm.sp500.fit=EMM(fit.returns.11118000,coef=rho.sp500.emm,
      control=EMM.control(n.burn=n.sp500.burn,n.sim=n.sp500.sim,
            initial.itmax=10,final.itmax=300,n.start=rep(5,2),
            tweak=c(.25,.5)),
      gensim.fn="Heston.QEM.gensim",
      gensim.language="SPLUS",
      gensim.aux=Heston.sp500.QEM.aux)

names(emm.sp500.fit$coef)=rho.sp500.emm.names
emm.sp500.fit
plot(emm.sp500.fit)

```

```

## use the auto fit function ##

emm.sp500.fit.auto=EMM(fit.returns.auto,coef=rho.emm,
  control=EMM.control(n.burn=n.burn,n.sim=n.sim,
    initial.itmax=10,final.itmax=300,n.start=rep(5,6),
    tweak=c(.25,.5,1,1.25,1.5,2)),
  gensim.fn="Heston.gensim",
  gensim.language="SPLUS",
  gensim.aux=sp500.emm.aux)

names(emm.sp500.fit.auto$coef)=rho.emm.names
emm.sp500.fit.auto
plot(emm.sp500.fit.auto)

```


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CHAPTER 5

IMPLIED MARKET PRICES OF RISKS AND MARKET RISK AVERSION

With the theoretical results about the equilibrium relationships between the MPRs and the MRA, as well as the numerical methods introduced in chapter 3 and chapter 4, we are now ready to extract the market prices of risks and risk aversion from the cross-sectional options data and the time series of stock index data. In general, the idea is to extract the values of market risk premium, which are the key concepts in this dissertation, by cross-sectional fitting. Namely, we have the following minimization problem:

$$\min_{\psi} \sum_{(K,T) \in \kappa} \left[C(K,T;\psi) - C^{obs}(K,T) \right]^2$$

where $C(K,T;\psi)$ is the theoretical call option prices calculated according to PDE (3.6) in chapter 3, $C^{obs}(K,T)$ is the observed market option prices, ψ is the set of parameters of interest. Implied market risk aversion is then easily computed by equation (3.15) in chapter 2. The parameter set ψ and the space set of strikes and time to maturities change depending on our specific goals. Recall that we are interested in three types of MRAs: individual MRAs across strikes, average MRA for the whole market, and theoretical MRA which should prevail in the market according to theory. Therefore the space set will include one strike for IMRA, all strikes for AMRA, and a mix of strikes for TMRA. On the other hand, we are also interested in comparing the behavior of MPRs between a restricted and an unrestricted model. In a restricted

model, we assume risk neutrality (hence fixing the value of market price of asset risk) and use model parameters estimated by EMM, so in this case the parameter set only includes the market price of volatility risk. In an unrestricted model, we allow all parameters to freely change to minimize the distance between the theoretical prices and market prices, so the parameter set includes all model parameters.

This chapter proceeds as follows. Section One describes the options data used in calibration. Section Two discusses the optimal algorithm to extract the volatility risk and the advantage and disadvantage of our cross-sectional fitting method. Section Three reports the extracted values of volatility risk and risk aversion under different scenarios, and discusses their implications. Section Four summarizes.

Section I – Data

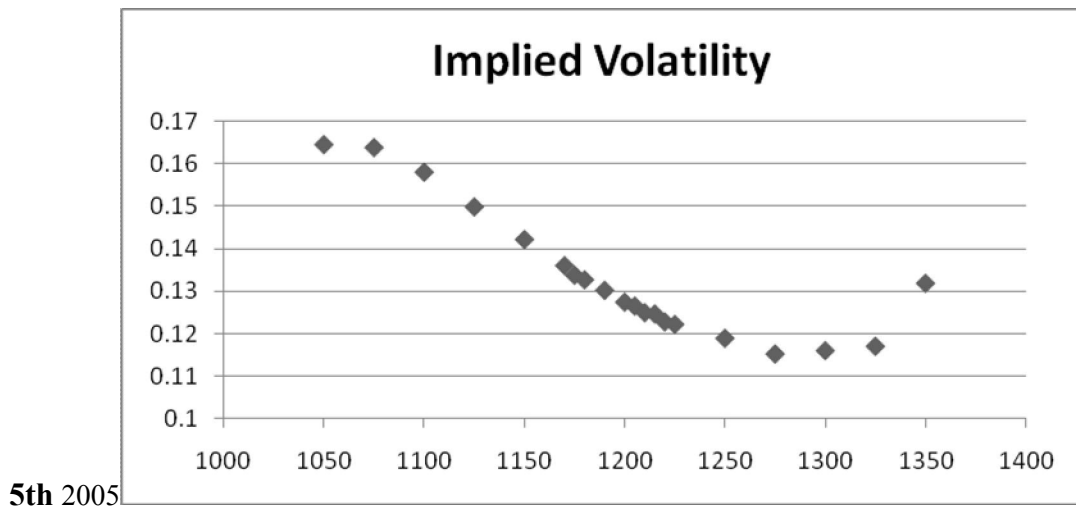
We use the S&P 500 index call and put options traded on CME on January 5th, 2005. The only reason we choose this particular date is that it is freely available on the website²³. We use the average of bid and ask prices. The time to maturity is 72 days. The raw data is attached in the appendices. Those options that were not traded on that day are discarded so we are left with 20 options for both calls and puts, corresponding to 20 strikes. To get a clearer picture of the data, implied volatilities are calculated. Since it is common, as in this case, that call option implied volatility and put option implied volatility are not the same, reflecting the market disagreed forward prices. Adopting the practitioner's approach, we use the put-call parity to back out the

²³ The data usage is by permission of Stephen Figlewski.

market implied forward prices, then use them to calculate implied volatilities, as shown in Table 5.1.

There have been a number of explanations for market implied volatility smiles. Some attribute the smile to the leverage effect which argues that a falling asset price causes the company's debt-equity ratio to decrease, hence increasing the asset price volatility. Further studies have shown, however, that the leverage effect can at most explain a small part of the volatility smile. Another popular argument is about market investors' risk attitudes. Since it is observed that out of the money put options are traded at premium this may suggest that market players are demanding put options to protect against market downturns. This argument, however, is built upon the assumption that the Black-Scholes model provides the "correct" prices. Now we all agree that the classic Black-Scholes model is at best a standard or reference for other more realistic pricing models to compare with, so in order for this argument to hold in our case it is natural to ask if the SVM predicted prices are high or low compared with market observed prices. Before, answering this question was equivalent to fitting the implied volatility smile to model produced volatility smiles, namely the cross-sectional fitting. But here we are considering a number of stochastic volatility markets completed with each put option with a certain strike, so we can now actually solve the inverse problem to back out the implied market price of volatility risk, hence risk aversion directly for each put option. If we still observe a higher risk aversion for OTM puts, then the risk aversion argument for explaining the implied volatility smile is robust at least up to stochastic volatility models. If we fail to observe such a pattern, i.e. an implied MRA smile, then either the argument fails or it indicates that we neglected some other risk factors that contribute to risk aversion. In a summary, we are trying to answer this question: *can risk aversion explain the market implied volatility smile?*

Table 7 - Implied Volatility Smile for S&P 500 Index Options Traded on January



Section II - Optimization Algorithms

We try several local minimization algorithms for the calibration purpose. One is the popular levenberg-marquardt algorithm; the other the Nelder-Mead algorithm. The performances are about the same in our case. We also tried global algorithms such as PIKAIA and Differential Evolution. Although these give out the global solution to our non-linear least squares problem the computing cost associated becomes unacceptably large. Hence we stick with the local optimization algorithms. In order to avoid getting stuck in a local minimum for each algorithm we try several different initial guesses by using random perturbation in our algorithms.

Although the computation is intensive cross-sectional fitting does have some advantages. First, the method does not need to estimate actual probability densities as does in Jackwerth (2000), Ait-Sahalia and Lo (2000). As noted in Rosenberg and Engel (2002), “(estimating) investor expectations about future return probabilities by smoothing a histogram of realized returns...are inconsistent with evidence from the

stochastic volatility modeling literature” and “their estimates are perhaps best interpreted as a measure of the average pricing kernel over the sample period”. Bliss and Panigirtzoglou (2004) also questioned that the assumption of time-invariant subjective densities made in these dissertations may be inconsistent with time-varying risk neutral densities. Second, the method does not assume any specifications of the pricing kernel as does in Bliss and Panigirtzoglou (2004) where they derived subjective densities augmented by assumed stationary risk aversion structures, i.e. by putting restrictions on the form of pricing kernel (power utility and exponential utility are considered) to allow for time-varying subjective densities. However, as criticized in Jackwerth (2004), “(the) particular form of the utility function...precluded finding a potential pricing kernel puzzle *a priori*”. Rosenberg and Engel (2002) also assume two forms of the pricing kernel: the power and the orthogonal polynomial form. It turns out that the latter produces the pricing kernel puzzle while the first does not. The implied MRA derived through the relation with EPK and MPR is also independent of the form of pricing kernel, as does in Bartunek and Chowdhury (1997) and others. Third, by using a stochastic volatility model the method is more general than that in Weron (2005) and Turvey (2006) while keeping the consistency with time-varying risk neutral densities. Fourth, the method for the first time allows us to simultaneously derive the market price of asset risk and the market price of volatility risk. Brennan and Schwartz (1979) also consider two kinds of MPRs, but by deliberately choosing the second state variable as the long term rate which is perfectly related to a tradable asset, the consol bond, their final formula for the empirical test does not involve the market price of long term rate risk. Hence they only extracted one MPR. Disadvantages of our method are first, we assume particular stochastic processes of the underlying state variables; second, it is computationally intensive.

Section III – Results

This section will examine the MPR and/or MRA with three approaches. First, we look at their behavior in the traditional Black-Scholes case where the martingale restriction assumption is relaxed; second, we consider the restricted model with stochastic volatility where all model parameters but the market price of volatility risk is predetermined through EMM; third, we consider the unrestricted model with stochastic volatility where all model parameters are allowed to change freely.

We first look at the empirical MPR and MRA under the benchmark Black-Scholes case. We follow the methods in Turvey (2006) to see if there are similar patterns in the options market. As shown in Figure 5.2 and Figure 5.3, both MPR for calls and for puts exhibit similar patterns for the live cattle market as examined in Turvey (2006). Implied MPR for calls frown across strikes and implied MPR for puts smile across strikes. A common trend for both is values are constantly decreasing across the strikes, indicating market rewards investors for taking out of the money risk. The MPR is positive and ranges from 2.3 to -17 for March call contract and from 50 to -25 for puts.

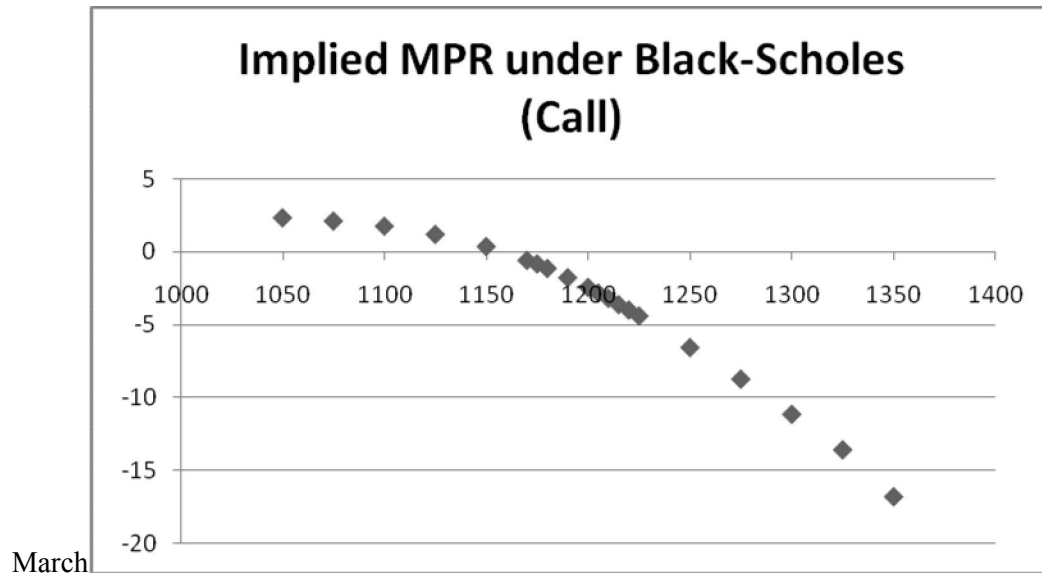


Figure 23 - Implied Market Price of Risk under Black-Scholes Model for

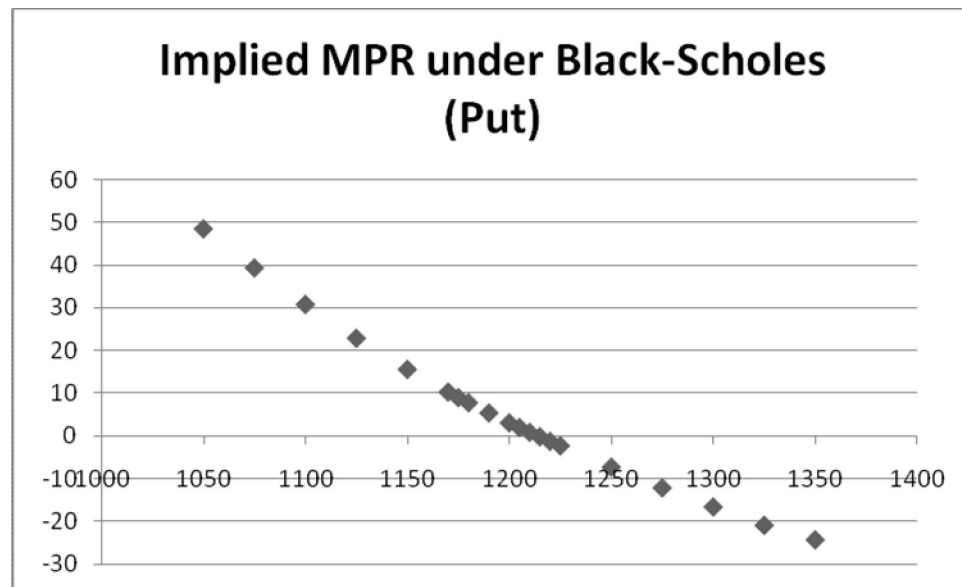


Figure 24 - Implied Market Price of Risk for Puts under Black-Scholes Model for March

Just as the implied volatility smile, the Figures above demonstrate the differences between the market prices and the model predicted prices. It is obvious that the market

overprices the out-of-the-money puts and out-of-the-money calls²⁴ for March, relative to the Black-Scholes model. This is consistent with Figure 5.1 the implied volatility smile and it is not surprising because if we keep the volatility constant across the strikes the only factor that reflects the price differences will be the market price of asset risk. As shown in chapter 3, the market asset risk is closely related to the risk aversion by $\gamma = \lambda / \sigma$ under Black-Scholes. Hence the Figure 5.3 indicates that the call option traders become more risk loving as the option moves from in the money to out the money, while Figure 5.4 shows that the put option traders become more risk loving as the option moves from out of the money puts to in the money puts. However, since the Black-Scholes model is now generally agreed to be too simplistic, the observations made above may not reflect the true values or trends for the MPRs and the MRA. As we adopt more realistic models such as the stochastic volatility models, we should expect that the values will make more sense. For example, we expect that the values of market prices of risks will decrease and hence the risk aversion will be small instead of being hundreds as in the Black-Scholes case.

Now we look at the behavior of MPR and MRA under restricted Heston model. See Figure 5.4 through Figure 5.7. We find that: first, the market price of volatility risk is in general quantitatively small compared with the market price of asset risk. The volatility risk ranges from 0 to 0.03 while the asset risk is fixed at 0.496, the market Sharpe ratio, both of which are indeed much smaller than under the Black-Scholes case. This also suggests that the largest contribution factor in the market risk aversion is the asset risk, not the volatility risk. Second, the value of implied risk aversion is consistent with what the literature reports, ranging from 2.2 to 4.3 which is much

²⁴ Note that in chapter 4 we have shown that the option price is decreasing with respect to the market price of asset risk for calls and increasing against the asset risk for puts.

lower than the equity premium literature but more common in the options market. Third, market players trading out of the money calls and out of the money puts are more risk averse than those trading near the money or at the money options, which is consistent with the existing risk aversion explanation of the implied volatility smile. Investors who are afraid of market melt down demand put options to protect themselves from falling asset prices, thus pushing up the market prices and causing volatility smiles. Mapped to implied risk aversion smile, we observe a higher risk aversion value for these groups of investors. This is clearly seen in Figure 5.7 where the MRA is highest for OTM puts. There are also investors who are risk seeking by purchasing in the money calls instead of outright stocks to take advantage of high leverage offered by the call options. This is reflected in Figure 5.5 where the MRA is lowest for ITM calls. Therefore, there is a one to one map between the market implied volatility smile and the market implied risk aversion smile. More importantly, we have confirmed that the risk aversion argument is valid even when we include stochastic volatility into our pricing model. It is shown that market players indeed exhibit risk aversion by paying higher prices for put protections and risk loving by demanding ITM calls. Lastly, note that in Figure 5.5 we have indicated the Sharpe ratio implied MRA of roughly 3 with the red straight line. Deviations from this red line reflect the parts due to the market price of volatility risk. For at the money calls this effect is negligible but becomes more significant as the contract moves farther out of the money.

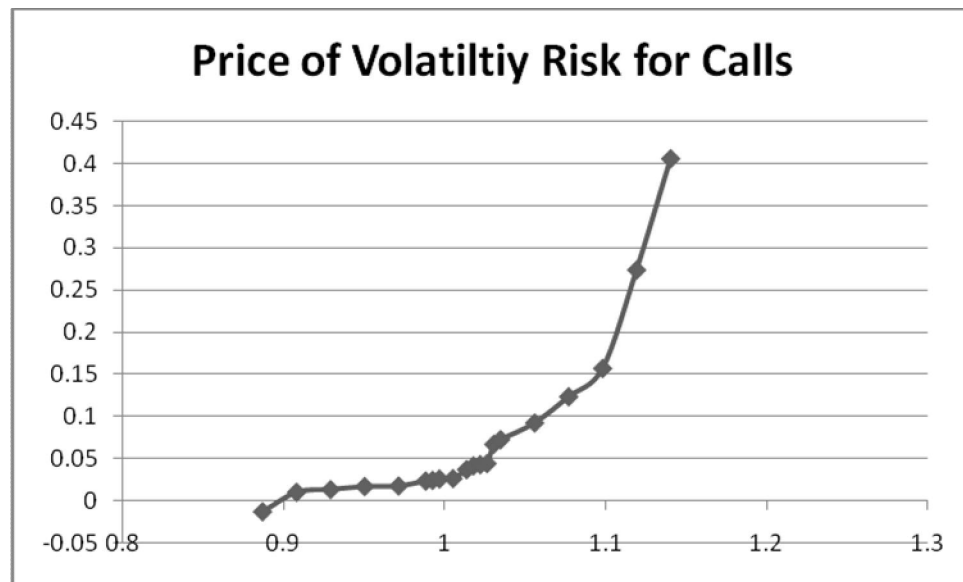


Figure 25 - Market Price of Volatility Risk under Heston SVM from Call Options

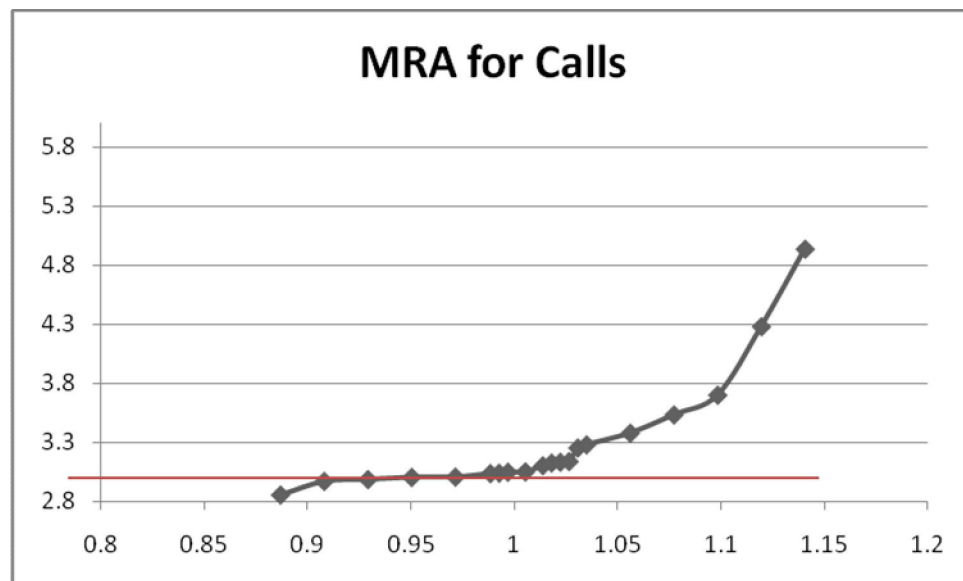


Figure 26 - Market Risk Aversion under Heston SVM from Call Options

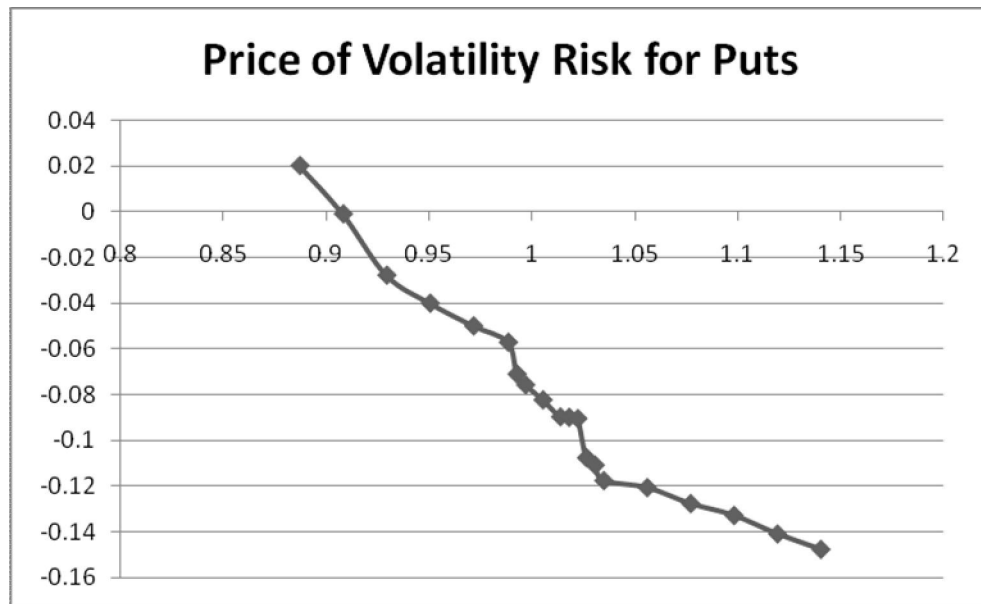


Figure 27 - Market Implied Volatility Risk from Put Options

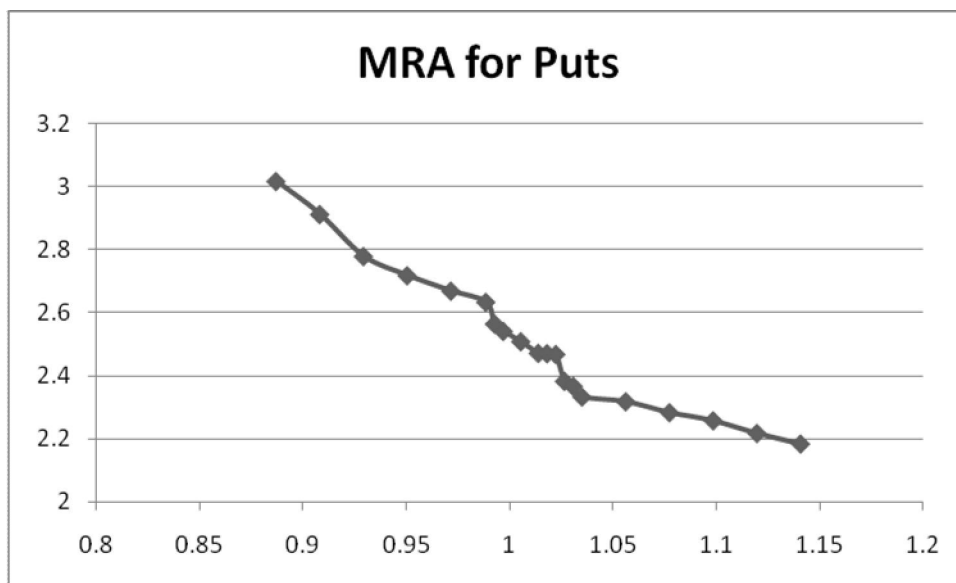


Figure 28 - Market Implied Risk Aversion from Put Options

The graph below shows the MRA extracted from individual stock options, namely the Volvo stock call options, in the research done by Benth, Groth and Lindberg (2010). The graph is the Figure 10 in their paper. One can see that our results for calls are similar to their findings.

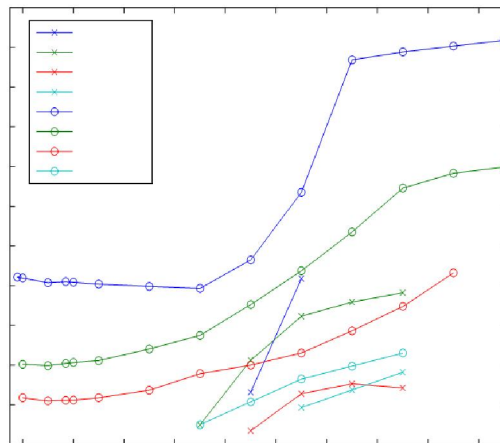


Figure 29 - MRA Extraction from Volvo Call Options in Benth, Groth and Lindberg (2010)

Next, we present empirical results for our unrestricted model where all model parameters are allowed to freely change to minimize the squared pricing errors. For this part, we used a different set of data; the CME traded S&P 500 index futures options, because they are freely available on the web. The data is attached in the appendix. There are six time to maturities and the number of strikes range from 62 to 30. Those option prices that give extremely high or low implied volatilities are omitted from our study as these indicate arbitrage opportunities. Figure 30 to Figure 35 show the major results for call options.

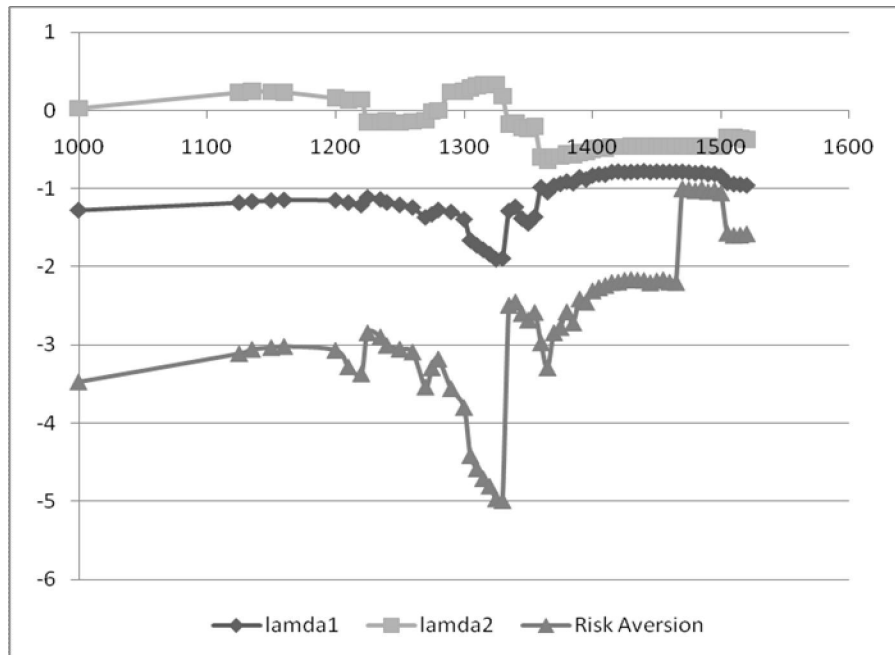


Figure 30 - Implied Market Prices of Risks and Risk Aversion for March Contracts

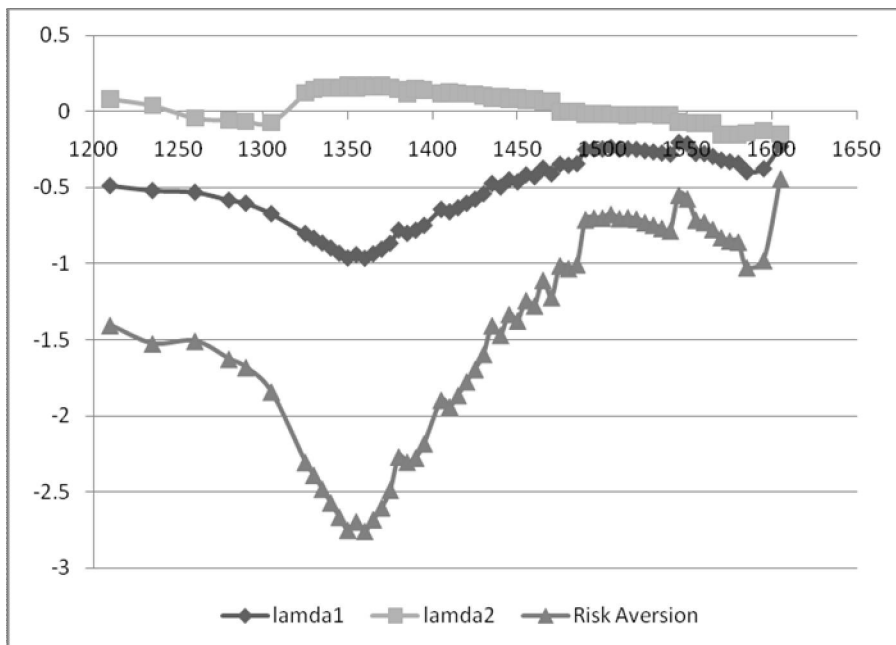


Figure 31 - Implied Market Prices of Risks and Risk Aversion for April Contracts

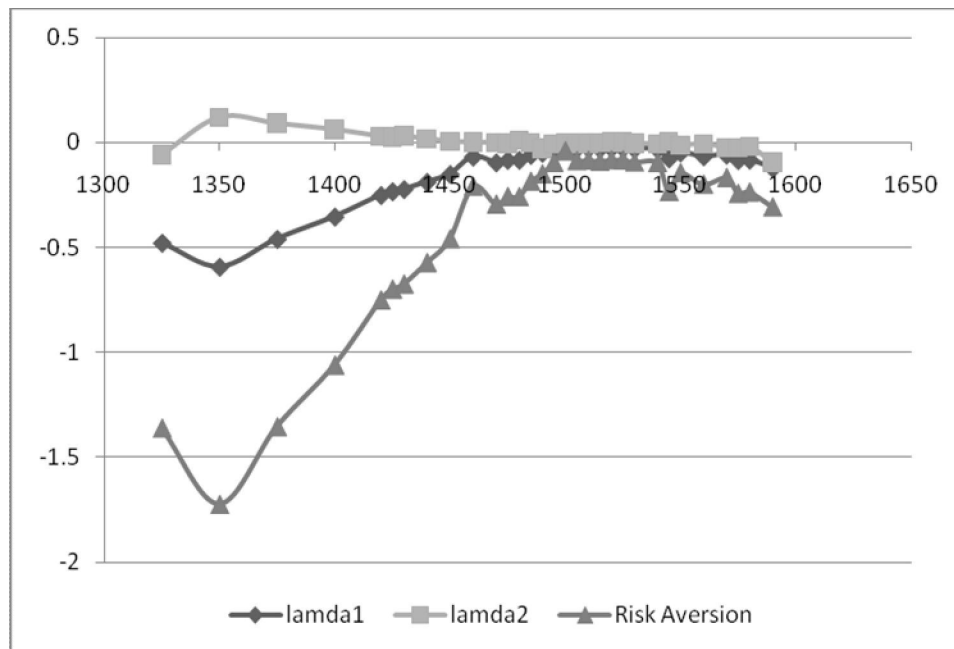


Figure 32 - Implied Market Prices of Risks and Risk Aversion for May Contracts

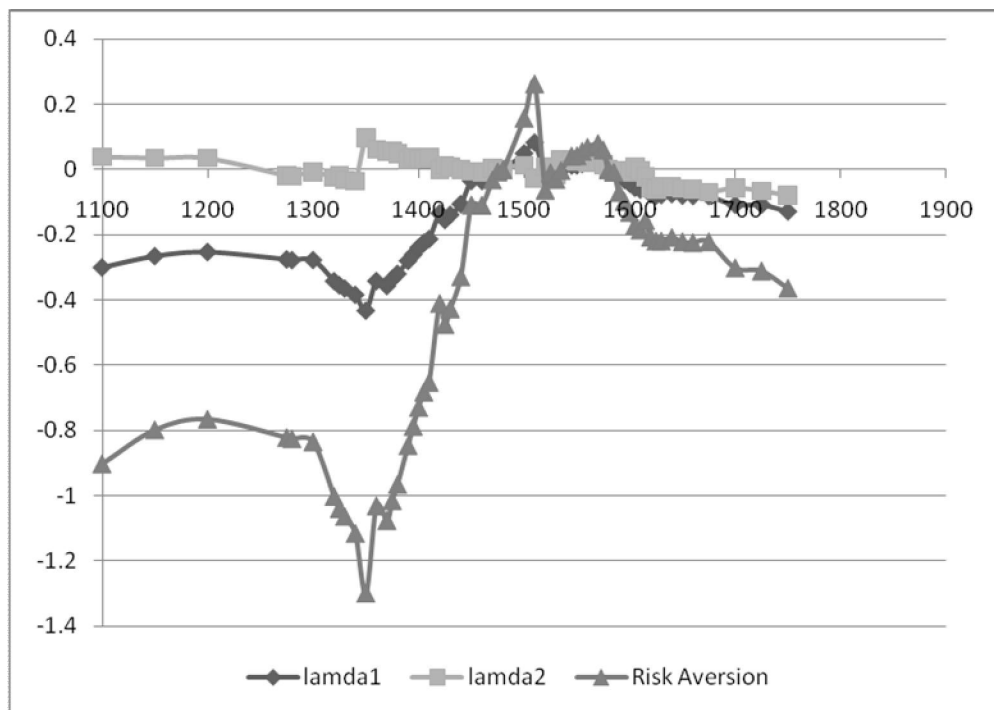


Figure 33 - Implied Market Prices of Risks and Risk Aversion for June Contracts

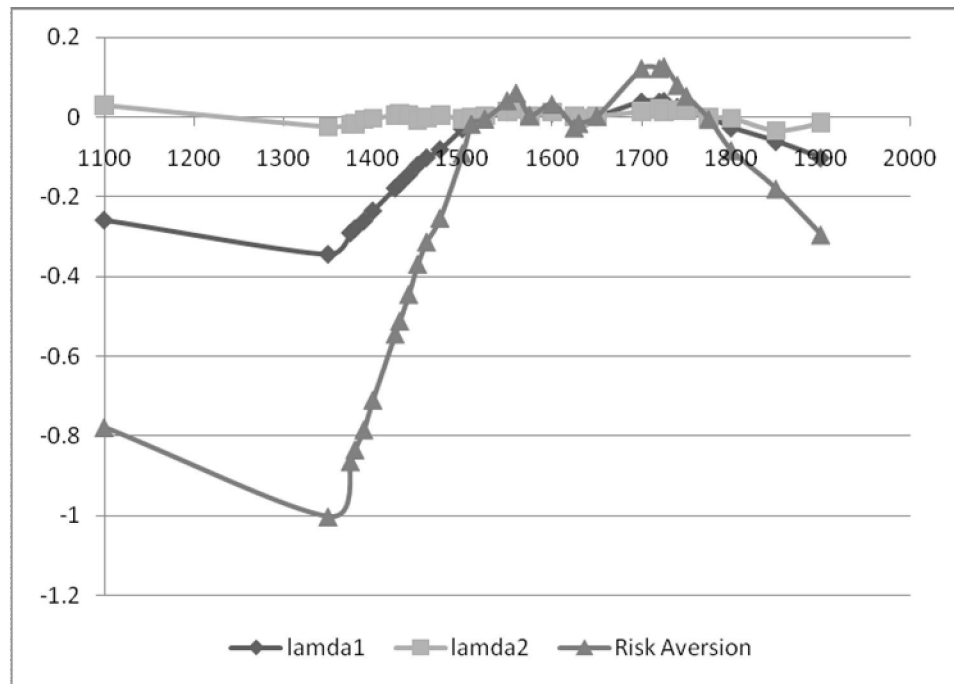


Figure 34 - Implied Market Prices of Risks and Risk Aversion for September Contracts

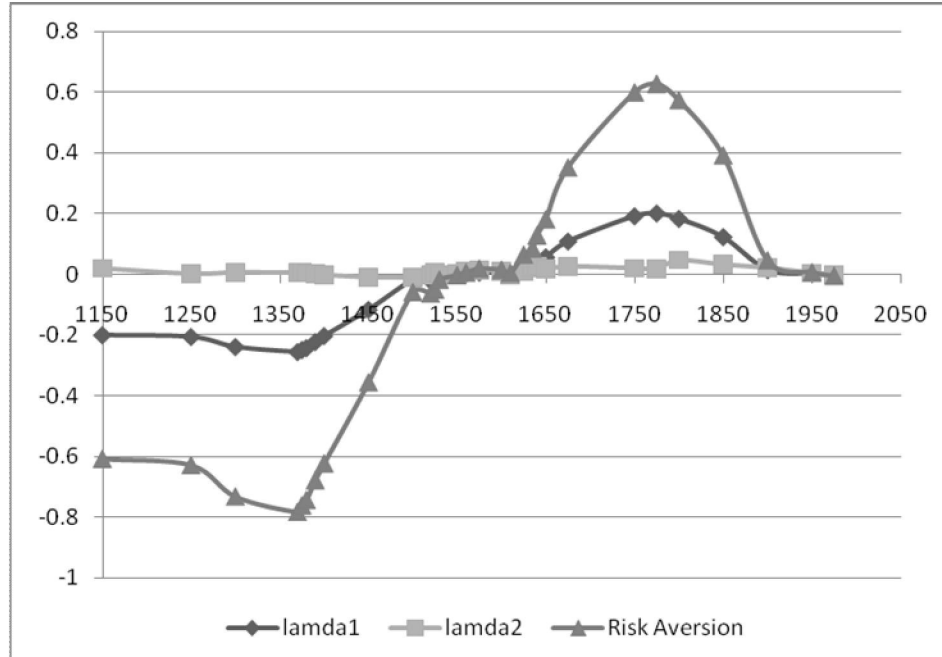


Figure 35 - Implied Market Prices of Risks and Risk Aversion for December Contracts

The unrestricted model provides richer and more interesting results than the restricted one since we now have one more dimension, the time to maturity. Several observations: first, as in the restricted model the market price of volatility risk is quantitatively small compared with the price of asset risk, again indicating that a large contribution of MRA comes from the asset risk, not the volatility risk. Second, the price of volatility risk and its variation both decreases toward zero, implying that holders of longer contracts care less about the volatility risk but more about the asset risk. Third, the size of implied MRA is also consistent with other studies in the literature and the findings in the restricted model, around 2-5 in general. Fourth, the MRA is generally increasing across strikes for call options, which is again consistent with that in the restricted model, reflecting the fact that market players exhibit risk loving by demanding ITM calls instead of outright stocks. Fifth, although MRAs are all increasing for each contract, across contracts the MRAs are becoming less and less, implying that investors with longer time horizons are less risk averse compared with those with shorter time horizons. Sixth, perhaps the most intriguing observation is that roughly the risk aversion hits the lowest point at the money and becomes higher in the tails, indicating that the market pricing kernels may be governed by different groups of investors across strikes. Lastly, the model parameters other than the two market prices of risks and initial volatility do not change very much across the strikes, verifying that the most relevant parameters for calculating MRA are the prices of risks and initial volatility, consistent with our theoretical results in chapter 2.

Section IV – Conclusion

This chapter presents the empirical result. Market prices of risks are extracted by least square optimization algorithms combined with a finite difference solution to the pricing equation of Heston SVM. Asset risk is the main factor to match the model-predicted price with the real market observed data. Asset risk exhibits smiling for near the money and deep in the money call options and frowning for out the money calls, which is a bit different from Turvey (2006). The scale of the smile/frown, however, is smaller compared to the Black-Scholes extracted risk prices. The reason is most likely because we use a more realistic stochastic volatility model. Volatility risk is quantitatively small and does not show any obvious pattern. Although the MRA does become flatter across the strikes under Heston compared with the Black-Scholes as we expected, we are not fully assured of that. The reason is because the empirical MRA is dependent upon many estimated parameters which are notoriously sensitive to both the selected time period of the data and the model under consideration. Therefore it is worthy to investigate in future research whether the empirical presented in this section is consistent and robust against different time series of data and across different models.

APPENDIX 5.1: S&P 500 INDEX OPTIONS TRADED ON JANUARY 5, 2005

S&P 500 Index Options Prices, Jan. 5, 2005

S&P 500 Index closing level, = 1183.74 Interest rate = 2.69

Option expiration: 3/18/2005 (72 days)

<u>Call</u>				<u>Put</u>			
<u>Strike</u>	<u>Bid</u>	<u>Ask</u>	<u>Average</u>	<u>Strike</u>	<u>Bid</u>	<u>Ask</u>	<u>Average</u>
500	N/A	N/A	N/A	500	0	0.05	0.025
550	N/A	N/A	N/A	550	0	0.05	0.025
600	N/A	N/A	N/A	600	0	0.05	0.025
700	N/A	N/A	N/A	700	0	0.1	0.05
750	N/A	N/A	N/A	750	0	0.15	0.075
800	N/A	N/A	N/A	800	0.1	0.2	0.15
825	N/A	N/A	N/A	825	0	0.25	0.125
850	N/A	N/A	N/A	850	0	0.5	0.25
900	N/A	N/A	N/A	900	0	0.5	0.25
925	N/A	N/A	N/A	925	0.2	0.7	0.45
950	N/A	N/A	N/A	950	0.5	1	0.75
975	N/A	N/A	N/A	975	0.85	1.35	1.1
995	N/A	N/A	N/A	995	1.3	1.8	1.55

1005	N/A	N/A	N/A	1005	1.5	2	1.75
1025	N/A	N/A	N/A	1025	2.05	2.75	2.4
1050	134.5	136.5	135.5	1050	3	3.5	3.25
1075	111.1	113.1	112.1	1075	4.5	5.3	4.9
1100	88.6	90.6	89.6	1100	6.8	7.8	7.3
1125	67.5	69.5	68.5	1125	10.1	11.5	10.8
1150	48.2	50.2	49.2	1150	15.6	17.2	16.4
1170	34.8	36.8	35.8	1170	21.7	23.7	22.7
1175	31.5	33.5	32.5	1175	23.5	25.5	24.5
1180	28.7	30.7	29.7	1180	25.6	27.6	26.6
1190	23.3	25.3	24.3	1190	30.3	32.3	31.3
1200	18.6	20.2	19.4	1200	35.6	37.6	36.6
1205	16.6	18.2	17.4	1205	38.4	40.4	39.4
1210	14.5	16.1	15.3	1210	41.4	43.4	42.4
1215	12.9	14.5	13.7	1215	44.6	46.6	45.6
1220	11.1	12.7	11.9	1220	47.7	49.7	48.7
1225	9.9	10.9	10.4	1225	51.4	53.4	52.4
1250	4.8	5.3	5.05	1250	70.7	72.7	71.7
1275	1.8	2.3	2.05	1275	92.8	94.8	93.8
1300	0.75	1	0.875	1300	116.4	118.4	117.4
1325	0.1	0.6	0.35	1325	140.8	142.8	141.8
1350	0.15	0.5	0.325	1350	165.5	167.5	166.5
1400	0	0.5	0.25	1400	N/A	N/A	N/A
1500	0	0.5	0.25	1500	N/A	N/A	N/A

APPENDIX 5.2: CLEANED DATA AND IMPLIED FORWARD PRICES

K	Call Average	iV for Calls	Put Average	iV for Puts	implied Forward
1050	135.5	0.164426788	3.25	0.164427	1182.963422
1075	112.1	0.163749416	4.9	0.163749	1182.77829
1100	89.6	0.157950649	7.3	0.157951	1182.743967
1125	68.5	0.149777465	10.8	0.149777	1183.011263
1150	49.2	0.142108577	16.4	0.142109	1182.97694
1170	35.8	0.135981865	22.7	0.135982	1183.170668
1175	32.5	0.133706637	24.5	0.133707	1183.043156
1180	29.7	0.132628692	26.6	0.132629	1183.116723
1190	24.3	0.130145092	31.3	0.130145	1182.962239
1200	19.4	0.12736633	36.6	0.127366	1182.707215
1205	17.4	0.126459507	39.4	0.12646	1182.881321
1210	15.3	0.124899259	42.4	0.124899	1182.753809
1215	13.7	0.124601943	45.6	0.124602	1182.927916
1220	11.9	0.1227082	48.7	0.122708	1183.001482
1225	10.4	0.12212764	52.4	0.122128	1182.773431
1250	5.05	0.118883637	71.7	0.118884	1182.990457
1275	2.05	0.115188565	93.8	0.115189	1182.755055
1300	0.875	0.115963633	117.4	0.115964	1182.846406
1325	0.35	0.11698308	141.8	0.116983	1182.786948
1350	0.325	0.131798937	166.5	0.131799	1182.928569

APPENDIX 5.3: RAW DATA FOR S&P 500 INDEX FUTURES OPTIONS

S&P500 Globex Futures Call Option Prices traded on CME on 03/03/2008

1000	331.3	N/A	N/A	N/A	N/A	N/A	N/A
1100	N/A	N/A	N/A	246.8	260	N/A	N/A
1125	206.7	N/A	N/A	N/A	N/A	N/A	N/A
1135	196.9	N/A	N/A	N/A	N/A	N/A	N/A
1150	182.2	N/A	N/A	204.2	N/A	232.9	N/A
1160	172.5	N/A	N/A	N/A	N/A	N/A	N/A
1200	134.2	146.3	N/A	163.9	N/A	N/A	N/A
1210	124.9	N/A	N/A	N/A	N/A	N/A	N/A
1220	115.9	N/A	N/A	N/A	N/A	N/A	N/A
1225	111.4	125.6	N/A	N/A	N/A	N/A	N/A
1235	102.6	N/A	N/A	N/A	N/A	N/A	N/A
1240	98.3	N/A	N/A	N/A	N/A	N/A	N/A
1250	89.9	106	N/A	N/A	N/A	163.2	N/A
1260	81.5	N/A	N/A	N/A	N/A	N/A	N/A
1270	73.3	N/A	N/A	N/A	N/A	N/A	N/A
1275	69.3	87.4	N/A	109.3	N/A	N/A	N/A
1280	63	83.9	N/A	105.9	N/A	N/A	N/A
1290	52.5	N/A	N/A	N/A	N/A	N/A	N/A
1300	45.2	70.2	N/A	89.3	N/A	132.1	N/A
1305	47.8	N/A	N/A	N/A	N/A	N/A	N/A
1310	44.5	N/A	N/A	N/A	N/A	N/A	N/A
1315	41.3	N/A	N/A	N/A	N/A	N/A	N/A
1320	38.3	57.4	N/A	80.5	N/A	N/A	N/A
1325	35.3	54.5	66.1	77.5	N/A	N/A	N/A
1330	28.3	51.5	N/A	74.6	N/A	N/A	N/A
1335	27.5	48.6	N/A	N/A	N/A	N/A	N/A
1340	20.8	45.9	N/A	68.8	N/A	N/A	N/A
1345	21.8	43.2	N/A	N/A	N/A	N/A	N/A
1350	19.5	37.3	52.2	63.3	86.3	N/A	N/A
1355	16	38	N/A	N/A	N/A	N/A	N/A
1360	13	35.6	N/A	54	N/A	N/A	N/A
1365	15.7	33.2	N/A	N/A	N/A	N/A	N/A
1370	11.5	30.9	N/A	53.1	N/A	93.7	N/A
1375	10.5	25.8	39.7	50.7	73.4	91.3	N/A

1380	8.4	26.6	N/A	48.4	71	88.8	N/A
1385	9.2	24.6	N/A	N/A	N/A	N/A	N/A
1390	6.1	22.7	N/A	43.9	66.2	84	N/A
1395	6.5	N/A	N/A	41.7	N/A	N/A	N/A
1400	4.3	17.3	29.2	39	61.6	79.3	N/A
1405	3.7	17.4	N/A	37.5	N/A	N/A	N/A
1410	3.3	15.9	N/A	35.5	N/A	N/A	N/A
1415	2.4	14.4	N/A	N/A	N/A	N/A	N/A
1420	1.8	13	22	29.5	N/A	N/A	N/A
1425	1.7	11.7	20.4	29.9	51.1	N/A	N/A
1430	1.2	9	18.9	28.2	49.2	N/A	N/A
1435	1.1	9.4	N/A	N/A	N/A	N/A	N/A
1440	0.75	7.5	16.2	24.9	45.4	N/A	N/A
1445	1.05	7.5	N/A	N/A	N/A	N/A	N/A
1450	0.6	5.7	13.7	20.2	41.8	58.3	N/A
1455	0.4	5.8	N/A	N/A	N/A	N/A	N/A
1460	0.5	4.1	9.5	19	38.3	N/A	N/A
1465	0.45	4.5	N/A	N/A	N/A	N/A	N/A
1470	0.3	3.2	9.5	16.5	N/A	N/A	N/A
1475	0.3	3	8.6	15.3	33.5	N/A	N/A
1480	0.25	2.5	7.8	14.2	N/A	N/A	N/A
1485	0.2	2.55	7	N/A	N/A	N/A	N/A
1490	0.15	2.15	6.3	N/A	N/A	N/A	N/A
1495	0.15	1.85	5.7	N/A	N/A	N/A	N/A
1500	0.2	1.2	4.5	10.2	26.4	38.8	N/A
1505	0.1	1.3	4.5	N/A	N/A	N/A	N/A
1510	0.1	1	4	8.6	23.8	N/A	N/A
1515	0.05	0.7	3.5	N/A	N/A	N/A	N/A
1520	0.1	0.7	3.1	7.2	N/A	35	N/A
1525	N/A	0.6	2.7	6.6	20.3	33.6	N/A
1530	N/A	0.5	2.35	6	N/A	32	N/A
1535	N/A	0.4	N/A	5.5	N/A	N/A	N/A
1540	N/A	0.35	1.5	N/A	N/A	N/A	N/A
1545	N/A	0.3	1.55	4.45	N/A	N/A	N/A
1550	N/A	0.25	1.3	4	15.2	27.1	N/A
1555	N/A	0.25	N/A	3.6	N/A	N/A	N/A
1560	N/A	0.2	1	3.25	13.4	24.8	N/A
1565	N/A	0.2	N/A	N/A	N/A	N/A	N/A
1570	N/A	0.15	0.75	2.6	N/A	N/A	N/A
1575	N/A	0.15	0.65	2.2	11	21.5	N/A

1580	N/A	0.1	0.55	2.05	N/A	N/A	N/A
1585	N/A	N/A	N/A	1.85	N/A	N/A	N/A
1590	N/A	0.1	0.8	1.65	N/A	N/A	N/A
1595	N/A	N/A	N/A	1.45	N/A	N/A	N/A
1600	N/A	0.05	N/A	1.3	7.7	16.5	N/A
1605	N/A	N/A	N/A	1.15	N/A	N/A	N/A
1610	N/A	N/A	N/A	1	N/A	15.2	N/A
1615	N/A	N/A	N/A	0.9	N/A	N/A	N/A
1620	N/A	N/A	N/A	0.8	N/A	N/A	N/A
1625	N/A	N/A	N/A	0.8	5.5	13.2	N/A
1630	N/A	N/A	N/A	0.6	5.1	N/A	N/A
1635	N/A	N/A	N/A	N/A	N/A	11.9	N/A
1640	N/A	N/A	N/A	0.5	N/A	11.3	N/A
1650	N/A	N/A	N/A	0.4	3.8	10.2	N/A
1660	N/A	N/A	N/A	0.3	N/A	N/A	N/A
1675	N/A	N/A	N/A	0.2	N/A	7.7	N/A
1700	N/A	N/A	N/A	0.1	1.65	5.8	N/A
1720	N/A	N/A	N/A	N/A	1.1	N/A	N/A
1725	N/A	N/A	N/A	0.05	1	N/A	N/A
1740	N/A	N/A	N/A	N/A	0.75	N/A	N/A
1750	N/A	N/A	N/A	0.05	0.6	3	N/A
1775	N/A	N/A	N/A	N/A	0.35	2.1	N/A
1800	N/A	N/A	N/A	N/A	0.2	1.4	N/A
1850	N/A	N/A	N/A	N/A	0.1	0.6	N/A
1900	N/A	N/A	N/A	N/A	0.05	0.25	N/A
1950	N/A	N/A	N/A	N/A	N/A	0.1	0.7
1975	N/A	N/A	N/A	N/A	N/A	0.05	N/A
1980	N/A	N/A	N/A	N/A	N/A	N/A	0.55

APPENDIX 5.4: VBA CODE FOR NELDER-MEAD ALGORITHM

Option Explicit

Option Base 1

' Nelder-Mead is an algorithm for searching local minimum

' To search for a GLOBAL minimum one has to use either simulated annealing or Bayesian MCMC method

' But if one has a "good" guess of the starting parameters then the Nelder-Mead can be applied

' To avoid being stuck in a local solution, one can just try a wide range of initial parameters and run the algorithm multiple times

Public Function NelderMead(fname As String, ByRef startParams As Variant) As Variant

Dim resmat() As Double

' Initiate various points

' x1 -- best point

' xn -- the last (n-th) point

' xw -- worst point

' xbar -- mean of the best n points

' xr -- reflection point

' xe -- expansion point

```

' xc -- outside expansion point

' xcc -- inside expansion point

Dim x1() As Double, xn() As Double, xw() As Double, xbar() _
As Double, xr() As Double, xe() As Double, xc() As Double, _
xcc() As Double

Dim funRes() As Double, passParams() As Double

Dim MAXFUN As Integer, TOL As Double

MAXFUN = 1000 ' Maximum number of Iterations

TOL = 0.0000000001 ' Tolerance level for the difference between best and worst
points

' Initialize coefficients used in reflection, expansion and shrink steps

Dim rho As Double, Xi As Double, gam As Double, sigma As Double, paramnum As
Double

rho = 1

Xi = 2

gam = 0.5

sigma = 0.5

paramnum = Application.Count(startParams)

ReDim resmat(paramnum + 1, paramnum + 1) As Double

ReDim x1(paramnum) As Double, xn(paramnum) As Double, _
xw(paramnum) As Double, xbar(paramnum) As Double, _

```

```

xr(paramnum) As Double, xe(paramnum) As Double, _
xc(paramnum) As Double, xcc(paramnum) As Double

ReDim funRes(paramnum + 1) As Double, passParams(paramnum) As Double

' initialize the first row of result matrix

resmat(1, 1) = Run(fname, startParams)

Dim i As Integer, j As Integer

For i = 1 To paramnum

    resmat(1, i + 1) = startParams(i)

Next i

' initialize the rest rows of result matrix

For j = 1 To paramnum

    For i = 1 To paramnum

        If (i = j) Then

            If (startParams(i) = 0) Then

                resmat(j + 1, i + 1) = 0.05

            Else

                resmat(j + 1, i + 1) = startParams(i) * 1.05

            End If

        Else

            resmat(j + 1, i + 1) = startParams(i)

        End If

    End If

Next j

```

```

    passParams(i) = resmat(j + 1, i + 1)

Next i

    resmat(j + 1, 1) = Run(fname, passParams)

Next j

' Hence now the first column of resmat provides the initial n points; see internal notes

'For j = 1 To paramnum

'  For i = 1 To paramnum

'    If (i = j) Then

'      resmat(j + 1, i + 1) = startParams(i) * 1.05

'    Else

'      resmat(j + 1, i + 1) = startParams(i)

'    End If

'    passParams(i) = resmat(j + 1, i + 1)

'  Next i

'  resmat(j + 1, 1) = Run(fname, passParams)

Next j

Dim lnum As Integer

For lnum = 1 To MAXFUN

    resmat = BubSortRows(resmat)

    If (Abs(resmat(1, 1) - resmat(paramnum + 1, 1)) < TOL) Then

        Exit For

```

```

End If

' Best point

Dim fl As Double

fl = resmat(1, 1)

For i = 1 To paramnum

    xl(i) = resmat(1, i + 1)

Next i

' n-th point

Dim fn As Double

fn = resmat(paramnum, 1)

For i = 1 To paramnum

    xn(i) = resmat(paramnum, i + 1)

Next i

' Worst point

Dim fw As Double

fw = resmat(paramnum + 1, 1)

For i = 1 To paramnum

    xw(i) = resmat(paramnum + 1, i + 1)

Next i

' Mean point

For i = 1 To paramnum

```

```

    xbar(i) = 0

    For j = 1 To paramnum

        xbar(i) = xbar(i) + resmat(j, i + 1)

    Next j

    xbar(i) = xbar(i) / paramnum

Next i

' Reflection point

For i = 1 To paramnum

    xr(i) = xbar(i) + rho * (xbar(i) - xw(i))

Next i

Dim fr As Double

fr = Run(fname, xr)

Dim shrink As Double

Dim newpoint() As Double

Dim newf As Double

shrink = 0 ' a dummy variable used to control whether we need to shrink to the
best point

If ((fr >= fl) And (fr < fn)) Then

    newpoint = xr

    newf = fr

ElseIf (fr < fl) Then ' in this case we expand the triangle

```

```

For i = 1 To paramnum

    
$$xe(i) = \bar{x}(i) + X_i * (x_r(i) - \bar{x}(i))$$


Next i

Dim fe As Double

fe = Run(fname, xe)

If (fe < fr) Then

    newpoint = xe

    newf = fe

Else

    newpoint = xr

    newf = fr

End If

ElseIf (fr >= fn) Then

    If ((fr >= fn) And (fr < fw)) Then ' in this case we do outside contraction

        For i = 1 To paramnum

            
$$xc(i) = \bar{x}(i) + \text{gam} * (x_r(i) - \bar{x}(i))$$


        Next i

        Dim fc As Double

        fc = Run(fname, xc)

        If (fc <= fr) Then

            newpoint = xc

```

```

        newf = fc

    Else

        shrink = 1

    End If

Else ' otherwise we do inside contraction

    For i = 1 To paramnum

        xcc(i) = xbar(i) - gam * (xbar(i) - xw(i))

    Next i

    Dim fcc As Double

    fcc = Run(fname, xcc)

    If (fcc < fw) Then

        newpoint = xcc

        newf = fcc

    Else

        shrink = 1

    End If

End If

End If

If (shrink = 1) Then ' if we shrink

    Dim scnt As Double

    For scnt = 2 To paramnum + 1

```



```

    For i = 1 To paramnum ' shrink each parameter

        resmat(scnt, i + 1) = x1(i) + sigma * (resmat(scnt, i + 1) - x1(i))

        passParams(i) = resmat(scnt, i + 1)

    Next i

    resmat(scnt, 1) = Run(fname, passParams)

Next scnt

Else ' i.e. if we don't shrink

    For i = 1 To paramnum

        resmat(paramnum + 1, i + 1) = newpoint(i) ' replace worst point with the
new point

    Next i

    resmat(paramnum + 1, 1) = newf ' update the corresponding function value

End If

Next lnum

If (lnum = MAXFUN + 1) Then

    ' MsgBox "Maximum Iteration (" & MAXFUN & ") exceeded"

End If

resmat = BubSortRows(resmat)

' we are interested in result stored in the 1st row

For i = 1 To paramnum + 1

    funRes(i) = resmat(1, i)

```

```
Next i  
  
NelderMead = funRes  
  
'PrintArray (NelderMead)  
  
End Function 'NelderMead
```

APPENDIX 5.5: VBA CODE FOR IMPLIED MPR UNDER BLACK-SCHOLES

Option Explicit

Option Base 1

Const MAXITER = 500

' This is the formula (7) and (19) as in Turvey(2006)

Public Function BSOptionValue(ByVal iopt As Double, ByVal S As Double, ByVal x
As Double, ByVal mu As Double, ByVal r As Double, ByVal q As Double, ByVal
lambda As Double, ByVal tyr As Double, ByVal sig As Double)

' Returns the Black-Scholes Value (iopt=1 for call, -1 for put; q=div yld)

' Uses BSDOne fn

' Uses BSDTwo fn

Dim eqt, ert, NDOne, NDTwo As Double

eqt = Exp((mu - lambda * sig - r - q) * tyr)

ert = Exp(-r * tyr)

If S > 0 And x > 0 And tyr > 0 And sig > 0 Then

 NDOne = Application.NormSDist(iopt * BSDOne(S, x, mu, lambda, q, tyr, sig))

 NDTwo = Application.NormSDist(iopt * BSDTwo(S, x, mu, lambda, q, tyr, sig))

 BSOptionValue = iopt * (S * eqt * NDOne - x * ert * NDTwo)

Else

BSOptionValue = 0

End If

End Function

Function BSDOne(S, x, mu, lambda, q, tyr, sig)

' Returns the Black-Scholes d1 value

BSDOne = (Log(S / x) + (mu - lambda * sig - q + 0.5 * sig ^ 2) * tyr) / (sig * Sqr(tyr))

End Function

Function BSDTwo(S, x, mu, lambda, q, tyr, sig)

' Returns the Black-Scholes d2 value

BSDTwo = (Log(S / x) + (mu - lambda * sig - q - 0.5 * sig ^ 2) * tyr) / (sig * Sqr(tyr))

End Function

Public Function BSVegaRN(S, x, mu, lambda, q, tyr, sig)

Dim d As Double

d = BSDOne(S, x, mu, lambda, q, tyr, sig)

BSVegaRN = S * Fz(d) * Sqr(tyr)

End Function

Function Fz(x)

Fz = Exp(-x ^ 2 / 2) / Sqr(2 * Application.Pi())

End Function

' We use a combination of bisection and newton-raphson to calculate implied volatility of options

Public Function BisNewtMPR(PutCall, S, K, mu, v, r, q, T, a, b, realC)

Dim EPS As Double

EPS = 0.00001

Dim lowCdif As Double, highCdif As Double, midP As Double, dxold As Double,

—

dx As Double, midCdif As Double, midCvega As Double, temp As Double, i As Double

lowCdif = realC - BSOptionValue(PutCall, S, K, mu, r, q, b, T, v)

highCdif = realC - BSOptionValue(PutCall, S, K, mu, r, q, a, T, v)

midP = 0.5 * (a + b)

dxold = (b - a)

dx = dxold

midCdif = realC - BSOptionValue(PutCall, S, K, mu, r, q, midP, T, v)

```

midCvega = BSVegaRN(S, K, mu, midP, q, T, v)

For i = 1 To MAXITER

    If (((midP - b) * midCvega - midCdif) * ((midP - a) * midCvega - midCdif) > 0)
    Or (Abs(2 * midCdif) > Abs(dxold * midCvega)) Then

        dxold = dx

        dx = 0.5 * (b - a)

        midP = a + dx

    Else

        dxold = dx

        dx = midCdif / midCvega

        temp = midP

        midP = midP - dx

    End If

    midCdif = realC - BSOOptionValue(PutCall, S, K, mu, r, q, midP, T, v)

    If (Abs(midCdif) < EPS) Then

        Exit For

    End If

    midCvega = BSVegaRN(S, K, mu, midP, q, T, v)

    If (midCdif < 0) Then

        b = midP

    Else

```

```
        a = midP  
    End If  
Next i  
BisNewtMPR = midP  
End Function
```

APPENDIX 5.6: VBA CODE FOR MONTE CARLO SIMULATION

Option Explicit

Option Base 1

' Declare all global variables

' Parameters

Public r As Double ' Interest Rate

Public d As Double ' Dividend Rate

Public T As Double ' Maturity Date

Public mu As Double ' Drift Rate for the underlying stock process

Public K As Double ' Strike Price

Public S As Double ' Spot Price

Public v As Double ' Initial Volatility

Public volofvol As Double ' Volatility of Voatility

Public kappa As Double ' Mean Reversion Rate

Public theta As Double ' Mean Reversion Level

Public rho As Double ' Correlation Coefficient between Two Brownian Motions

Public lambda1 As Double ' Market Price of Asset Risk

Public lambda2 As Double ' Market Price of Volatility Risk

Public optionPrice As Double ' Store the calculated option price

Public numStrikes As Double ' Number of strikes in the data

Public numMaturities As Double ' Number of maturity dates in the data

Public numOfObs As Double ' Cross-section number of observations


```

Public Function MonteCarlo(ByVal iter As Double, ByVal T As Double, ByVal K As
Double) As Double
Dim e As Double, ee As Double, e2 As Double
Dim logSt As Double, deltat As Double, curS As Double, curV As Double, _
sqrootVt As Double, numOfSteps As Integer, optionPrice As Double
Dim spotArray() As Double
deltat = 1 / 365
ReDim spotArray(1 To Int(T * 365)) As Double
With ThisWorkbook.Worksheets("parameters")
    r = .Range("interestrate").Value
    d = .Range("dividendrate").Value
    S = .Range("spotprice").Value
    mu = .Range("driftrate").Value
    volofvol = .Range("volofvol").Value
    kappa = .Range("meanreversionrate").Value
    theta = .Range("meanreversionlevel").Value
    rho = .Range("corr").Value
    lambda1 = .Range("assetrisk").Value
    'lambda2 = .Range("volrisk").Value
    v = .Range("initialvol").Value
End With
optionPrice = 0
Dim i As Integer, j As Integer
For j = 1 To iter
    curS = S
    curV = v

```

```

logSt = Log(curS)
sqrootVt = Sqr(curV)
For i = 1 To (Int(T * 365))
    spotArray(i) = curS
    e = Moro_NormSInv(genrand_real3())
    ee = Moro_NormSInv(genrand_real3())
    e2 = rho * e + Sqr(1 - rho ^ 2) * ee
    'using euler scheme for the underlying stock process
    logSt = logSt + ((mu - lambda1 * Sqr(curV)) - 0.5 * curV - d) * deltat + Sqr(curV)
    * Sqr(deltat) * e

    curS = Exp(logSt)
    'using milstein scheme for the variance process

sqrootVt = sqrootVt + (kappa * (theta - sqrootVt ^ 2) - 0.25 * volofvol ^ 2 - (rho *
lambda1 + Sqr(1 - rho ^ 2) * lambda2) * sqrootVt * volofvol) * deltat / (2 * sqrootVt)
+ 0.5 * volofvol * Sqr(deltat) * e2

    curV = sqrootVt ^ 2
Next i
optionPrice = optionPrice + Exp(-T * r) *
Application.Max(spotArray(UBound(spotArray)) - K, 0)

Next j

optionPrice = optionPrice / iter
MonteCarlo = optionPrice
ThisWorkbook.ActiveSheet.Range("A1").Value = optionPrice
End Function 'Monte Carlo

```

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CHAPTER 6

CONCLUSION

In chapter 1 I identified the following objectives that this dissertation tries to achieve: first, understand the meaning of the concept of market price of risk; second, derive theoretical relationships between the market pricing kernel, market prices of risks and market risk aversion; third, study the empirical behavior of market prices of risks and risk aversion and examine their relations with the pricing kernel puzzle and option implied smiles.

Market price of risk is a very important yet also very confusing concept in modern finance literature. Traditionally it is defined to be the excess return of an asset per unit of volatility. However, in incomplete markets where nontradable assets are either the underlying or part of the risk factors, the associated market prices of risks are hard to interpret in their usual meanings. Specifically, in a stochastic volatility model we have two MPRs: the price of asset risk and the price of volatility risk. Many previous studies do not differentiate between them and treat a correlation weighted average of these two prices as the volatility risk premium. As shown in Chapter 2, however, the two risk prices have to be separated to consider the MRA. Hence in this dissertation I explicitly define these two MPRs and extract them empirically. A thorough sensitivity analysis of the option prices and implied volatilities with respect to the MPRs with the aid of finite difference method to solve the pricing partial differential equation indicates that both are important factors in determining the implied volatility smiles.

As much as we are aware of, there have been very few, if any, studies done in this respect.

In chapter 2 a new interpretation of the concept of MPR that is suitable for all kinds of market prices of risks is rendered by considering the relationship between market pricing kernel and market prices of risks when the economy considered is in equilibrium. Solving the representative agent's portfolio optimization problem using dynamic programming techniques I find that in a stochastic volatility model completed by put options and a pure exchange economy where the representative agent exists, an MPR is proportional to the derivative of the market pricing kernel, the present value of Arrow-Debreu security price, with respect to the corresponding risk factor. This new interpretation applies to nontradable risk factors such as stochastic volatility. A negative market price of volatility risk, for example, implies that the market representative agent values his/her wealth more in a state with higher asset volatility than in a state with lower volatility. In contrast, in a state with higher underlying asset price, the agent assigns lower value to the marginal utility increments, hence producing a positive market price of asset risk. Intuitively, most market investors behave as predicted by economic theory that marginal utility declines in wealthy state, hence exhibiting risk aversion. However, their risk attitudes in a volatile compared with a stable state are not clear at this point. The conceptual reinterpretation of the market price of volatility risk in this dissertation allows us to have a peek of this kind of behavior. Another contribution of this quantitative relation between the MPK and MPRs is that when it is substituted into the usual risk neutral pricing equation it gives back the well known Garman (1976) fundamental pricing equation. The latter does not rely on any assumptions on the form of the MPK, hence is more general than risk

neutral valuation. Hence indirectly I have shown the validity of Garman's pricing equation. To my best knowledge, this is the only paper to do so.

While the equilibrium relation between the pricing kernel and the market prices of risks provides additional insights of what the market prices of risks actually mean, the equilibrium relation between market prices of risks and risk aversion derived also in chapter 2 by considering a habit formation utility lays a theoretical foundation for empirically extracting the market risk aversion from real market data. I show for the first time in literature that in equilibrium the market risk aversion can be explicitly written in terms of the MPRs and the marginal utility due to habit level. Under habit formation, the instantaneous market risk aversion is not only dependent on the current pricing kernel and its derivative with respect to the consumption, but also relies on the differential between the future cumulative disutility from the habit level and its expectation. If the differential is negative, i.e. the future cumulative disutility at time t is less than its expectation, then the agent weighs the disutility more than the current marginal utility of consumption hence exhibits higher risk aversion. On the other hand, if the differential is positive, then the agent weighs more on the current marginal utility than on the future disutility hence exhibits less risk aversion. This property permits the market risk aversion under habit formation to vary randomly over time as the differential changes. More importantly, it poses a potential theoretical challenge to the robustness of the documented pricing kernel puzzle and others following the same approach. If the empirical risk aversion is extracted based on the time-separable preference assumption, then the pricing kernel puzzle – locally increasing marginal rate of substitution - may be just a disguised phenomenon of time varying risk aversion. In fact, under time-separable preference case, the MRA is instantaneously a linear combination of the MPRs. The first component is the traditional Black-Scholes

risk aversion due to price of asset risk, the second component coming from the price of volatility risk. In words, the MRA needs to be adjusted for the second risk factor – stochastic volatility – to get the correct value. This is exactly because the asset risk involves not only the sensitivity of the MPK with respect to the underlying asset but also the sensitivity of the MPK with respect to the volatility, while the MRA only relates to the former.

Recall that the model under consideration is a stochastic volatility model completed by put options. The key point here is that each traded European put option will complete the market, hence correspond to a representative agent with a certain market risk aversion. A natural follow-up is then to examine how MRAs look like across the strikes. This is our third objective.

To realize this objective, I first implemented a finite difference method in Excel/VBA in chapter 3 to price both European and American options under stochastic volatility models. The efficacy of the algorithms is verified by comparison with reports from previous studies in the option pricing literature. Although the procedures and ideas of implementing the finite difference method are identified in Duffy (2005), I write most of the code in VBA by myself.

When it comes to extract the MPRs and the MRA, I ignore the habit formation in this dissertation because that involves making assumptions on the utility function. Has time permitted, a calibration of the model allowing habit formation will be very helpful in explaining the pricing kernel puzzle. Then I am left with two choices of models: one is the unrestricted model where all model parameters are allowed to change to minimize the pricing errors between the theoretical price and the market

price; the other is the restricted model where only the price of volatility risk is allowed to change while all others are estimated from the time series of the underlying asset. Although the first approach has little economic sense since some model parameters such as the drift rate, the mean reversion level, the mean reversion rate have no reason to vary across the strikes, I implement it anyway for three reasons. First, this type of cross-sectional fitting is common in practice as it is easy to implement. Second, previous sensitivity analysis has suggested that only some model parameters, which happen to be of my interest, including the MPRs, the correlations and the initial volatility, are important in explaining the pricing error. So even I allow all parameters to float, I expect only those interested variables to fluctuate significantly across strikes. In order to do this checking on the unrestricted model, however, I need to estimate those parameters first. I choose the efficient method of moments because it has been widely used in modern literature and very successful in terms of parameter estimations. Chapter 4 introduces this method and presents the model parameter estimates using S-plus software. A little twist here is that the EMM is a simulation based statistical test which requires discretizing the continuous time Heston stochastic volatility model and normal discretization schemes such as Euler and Milstein have positive probability of producing negative volatilities. To avoid this problem, I choose the QEM algorithm which has been proved in practice very efficient for most combinations of the model parameters. The combination of QEM and EMM is, as far as I know by the time I write this dissertation, is the first in literature. The third reason I am interested in an unrestricted model is that I want to have some idea of the validity of the risk neutrality assumption, i.e. setting the asset price of risk to be the underlying asset's Sharpe ratio. For the lack of data, I did not regress the values of asset price of risk against the Sharpe ratio, but it is certainly doable in future research.

The reports in chapter 5 about the unrestricted model using S&P 500 index futures options are quite revealing. The price of volatility risk is quantitatively small for all strikes and time to maturities, relative to the price of asset risk. This suggests that the largest contribution to market risk aversion comes from the asset risk, not the volatility risk. Moreover, the volatility risk premium diminishes toward zero and the variation reduces with the time to maturity, indicating that the asset risk premium dominance becomes stronger.

For the restricted model, both the sign and the shape of implied market risk aversions across strikes are important. If the sign becomes negative for all or part of the moneyness, the pricing kernel puzzle can potentially be attributed to the stochastic volatility. And I can actually tell if the negativity of MRA is due to the asset price of risk or the volatility risk or both. Using S&P 500 index options I find that the implied MRA is positive for all strikes, indicating that stochastic volatility alone cannot explain the pricing kernel puzzle. This makes the hypothesis test with habit formation more important. In terms of the shapes, I find that the implied MRA is constantly decreasing across moneyness, exhibiting a smile, which implies that for deep in the money put options, investors are very risk averse. This is consistent with the observed implied volatility smile where investors are generally willing to pay a high premium for deep in the money puts for risk protection. The smiling shape is also consistent with the heterogeneous investor theory that the market pricing kernel is dominated by investors with extreme risk attitudes. This observation reminds us of the work of Sprenkle (1967). In his work, Sprenkle derived a warrant pricing formula that explicitly has market risk aversion as one of the parameters. I suspect that his model can well be used to explain the later found option volatility smiles.

I have shown that stochastic volatility alone cannot reproduce the pricing kernel puzzle, and there are probably other risk factors contributing to the risk aversion. Hence future work could explore further by extracting empirical market prices of risks and market risk aversion in models more general than Heston. For example, in light of the findings in Chernov & Ghysels (2000) a two volatility factor stochastic volatility model can be considered. Since our result of risk aversion is suitable for multi-state-variable models, adding extra volatility factors should not pose any problem. One can also consider a stochastic volatility model with jump process since the jump risk has been shown to be significant. It will be very intriguing to compare the empirical risk aversions between Black-Scholes, Heston, SVJ and 2-factor SVMs. Another task will be to calibrate the model under habit formation and see if including habit levels can reproduce pricing kernel puzzle. On the other hand, we can also try different sets of options data, e.g. index options for other markets such as FTSE, to see if our results are consistent and reliable. And as aforementioned, different periods of stock index data may result in different estimates of model parameters which may lead to different values of market prices of risks and risk aversion. Future research should investigate the robustness of results presented in this dissertation.